KINETIC FORMULATION FOR A PARABOLIC CONSERVATION LAW. APPLICATION TO HOMOGENIZATION.

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Abstract.

We derive a kinetic formulation for the parabolic scalar conservation law $\partial_t u + \mathrm{div}_y A(y,u) - \Delta_y u = 0$. This allows us to define a weaker notion of solutions in L^1, which is enough to recover the L^1 contraction principle. We also apply this kinetic formulation to a homogenization problem studied in a previous paper; namely, we prove that the kinetic solution u^ε of $\partial_t u^\varepsilon + \mathrm{div}_x A\left(x/\varepsilon, u^\varepsilon\right) - \varepsilon \Delta_x u^\varepsilon = 0$ behaves in L^1_{loc} as $v\left(x/\varepsilon, \bar{u}(t,x)\right)$, where v is the solution of a cell problem and $\bar{u}$ the solution of the homogenized problem.

Key words. Scalar conservation law, kinetic formulation, homogenization

AMS subject classifications. 35K55, 35B27

1. Introduction. This paper is devoted to the study of the solution $u \in \mathcal{C}([0,\infty), L^1(Y)) \cap L^2_{\mathrm{loc}}([0,\infty), H^1_{\mathrm{per}}(Y)) \cap L^\infty_{\mathrm{loc}}([0,\infty)\times Y)$ of the equation

$$\begin{cases} \partial_t u(t,y) + \mathrm{div}_y A(y, u(t,y)) - \Delta_y u(t,y) = 0, \quad t>0,\ y\in Y \\ u(t=0,y) = u_0(y), \end{cases} \tag{1.1}$$

where $Y = [0,1]^N$ is the N-dimensional torus; $A = A(y,v) \in \mathbb{R}^N$, $y \in Y$, $v \in \mathbb{R}$ is a given N-dimensional flux, periodic in the space variable y.

In [5], a kinetic formulation was derived for such heterogeneous conservation laws (in fact, this work was achieved for hyperbolic laws, but it can be generalized to parabolic laws with no difficulty), based on the previous papers of P.-L. Lions, B. Perthame and E. Tadmor concerning hyperbolic homogeneous conservation laws (see [14], [13], [18], [16], and the general presentation in [17]). However, this formulation is not entirely satisfactory : indeed, it is based on the comparison between the solution $u(t,y)$ of the conservation law and the constants via the function $\mathbf{1}_{v<u(t,y)}$, where v is an additional fluctuation variable. But the constants, which happen to be stationary solutions of homogeneous conservation laws, no longer play a special role in the context of heterogeneous conservation laws. Hence, our goal in this article is to derive a kinetic formulation based on the study of the stationary solutions of (1.1). Let us mention a related work of E. Audusse and B. Perthame [2], which defines a notion of entropy solution which is not based on Kruzkhov's inequalities, but rather on the comparison with special stationary solutions, and which is sufficient to derive the L^1 contraction principle.

Let us precise a few notations which will be used later on : if $\mathcal{C}^\infty_{\mathrm{per}}(Y)$ denotes the

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space of Y-periodic functions in $\mathcal{C}^\infty(\mathbb{R}^N)$, then

$$W^{k,p}_{\text{per}}(Y) := \overline{\mathcal{C}^\infty_{\text{per}}(Y)}^{W^{k,p}(Y)},$$

$$W^{1,\infty}_{\text{per,loc}}(Y\times\mathbb{R}) := \{u = u(y,v) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^{N+1}), u \text{ is } Y-\text{periodic in } y\},$$

$$\mathcal{D}_{\text{per}}([0,\infty)\times Y\times\mathbb{R}) := \{u = u(t,y,v) \in \mathcal{C}^\infty([0,\infty)\times\mathbb{R}^{N+1}),$$
$$u \text{ is periodic in } y \text{ and } \exists R>0,\ u(t,y,v)=0 \text{ if } t+|v|\geq R\},$$

$$\langle v\rangle := \frac{1}{|Y|}\int_Y v(y)\,dy \quad \forall v\in L^1(Y).$$

First, let us recall a few results on the stationary solutions of (1.1), which were studied in [3]:

Proposition 1.1. *Let* $A = A(y,v) \in W^{1,\infty}_{per,loc}(Y\times\mathbb{R})^N$.
Let $a_i(y,v) := \partial_v A_i(y,v)$, $1\leq i\leq N$, $b(y,v) := \text{div}_y A(y,v) \in L^\infty_{loc}(\mathbb{R}^{N+1})$. *Assume that there exist real numbers* $C_0>0$, $m\in[0,\infty)$, $n\in[0,\frac{N+2}{N-2})$ *when* $N\geq 3$, *such that for all* $(y,p)\in Y\times\mathbb{R}$

$$|a_i(y,p)| \leq C_0\left(1+|p|^m\right) \quad \forall\ 1\leq i\leq N, \tag{1.2}$$

$$|b(y,p)| \leq C_0\left(1+|p|^n\right). \tag{1.3}$$

Assume as well that the couple (m,n) *satisfies at least one of the following conditions*

$$m = 0 \tag{1.4}$$

$$\text{or}\quad 0\leq n<1 \tag{1.5}$$

$$\text{or}\quad n<\min\left(\frac{N+2}{N},2\right)\ \text{and}\ \exists p_0\in\mathbb{R},\ \forall y\in Y\ b(y,p_0)=0. \tag{1.6}$$

Then for all $p\in\mathbb{R}$, *there exists a unique solution* $v(\cdot,p)\in H^1_{per}(Y)$ *of the equation*

$$-\Delta_y v(y,p) + \text{div}_y A(y,v(y,p)) = 0, \quad \langle v(\cdot,p)\rangle = p. \tag{1.7}$$

For all $p\in\mathbb{R}$, $v(\cdot,p)$ *belongs to* $W^{2,q}_{per}(Y)$ *for all* $1<q<+\infty$ *and satisfies the following a priori estimate : for all* $R>0$, *there exists a constant* $C_R>0$ *depending only on* N, Y, C_0, m, n, q, p_0 *and* R, *such that*

$$||v(\cdot,p)||_{W^{2,q}(Y)} \leq C_R \quad \forall p\in\mathbb{R},\ |p|\leq R. \tag{1.8}$$

Moreover, for all $p\in\mathbb{R}$, $\partial_p v(\cdot,p)\in H^1_{per}(Y)$ *and is a solution of*

$$-\Delta_y\frac{\partial v}{\partial p} + \text{div}_y\left[a(y,v(y,p))\frac{\partial v}{\partial p}\right] = 0, \quad \left\langle\frac{\partial v}{\partial p}\right\rangle = 1. \tag{1.9}$$

And for all $R>0$, *there exists* $\alpha>0$ *depending only on* N, Y, C_0, m, n, q, p_0 *and* R, *such that for all* $(y,p)\in Y\times(-R,R)$,

$$\frac{\partial v}{\partial p}(y,p)\geq\alpha>0.$$

Equation (1.7) is also called “cell problem”, on account of its significance in homogenization problems.

Following the idea of E. Audusse and B. Perthame (see [2]), we now give a notion of entropy solution for equation (1.1) based on the comparison with stationary solutions :

Definition 1.2. *Assume the hypotheses of proposition 1.1 are satisfied.*

Let $u \in \mathcal{C}([0,\infty), L^1(Y)) \cap L^2_{loc}([0,\infty), H^1_{per}(Y)) \cap L^\infty_{loc}([0,\infty)\times Y)$ *be a solution of* (1.1)*. We say that* u *is an entropy solution of* (1.1) *if* u *satisfies the inequality*

$$\partial_t (u(t,y)-v(y,p))_+ + \mathrm{div}_y\left[\mathbf{1}_{u>v(y,p)}(A(y,u)-A(y,v(y,p)))\right] - \Delta_y (u(t,y)-v(y,p))_+ \leq 0 \tag{1.10}$$

for all $p\in\mathbb{R}$ *and in the sense of distributions on* $[0,\infty)\times Y$.

Notice that this notion of entropy solution is different (at least in its formulation) from the one of Kruzkhov, since the latter is based on the comparison with constants. However, inequality (1.10) was known by Kruzkhov, since it can be considered as a particular case of the comparison principle (notice that $v(y,p)$ is a stationary solution of (1.1)). It will be proved in the second section, under suitable regularity assumptions on the flux function A, that all solutions of (1.1) are entropy solutions in the sense of definition 1.2.

Let us mention here an important application of inequality (1.10) and of the kinetic formulation which follows from (1.10) : we give in this paper another proof for a homogenization result proved in [3], which we recall here for the reader's convenience :

Proposition 1.3. *Assume that* $A \in W^{1,\infty}_{per,loc}(\mathbb{R}^{N+1})^N$ *satisfies the assumptions of proposition 1.1, and that* $\partial_{y_j} a_i \in L^1_{loc}(\mathbb{R}^{N+1})$, $\partial_v a_i \in L^1_{loc}(\mathbb{R}^{N+1})$ *for* $1\leq i\leq N+1$, $1\leq j\leq N$.

For $\varepsilon>0$, *let* $v^\varepsilon \in L^\infty_{loc}([0,\infty)\times\mathbb{R}^N)\cap\mathcal{C}([0,\infty), L^1_{loc}(\mathbb{R}^N))\cap L^2_{loc}([0,\infty), H^1_{loc}(\mathbb{R}^N))$ *be a solution of the parabolic scalar conservation law :*

$$\frac{\partial v^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, v^\varepsilon(t,x)\right) - \varepsilon\Delta_x v^\varepsilon = 0 \quad t\geq 0,\ x\in\mathbb{R}^N, \tag{1.11}$$

$$v^\varepsilon(t=0) = v_0\left(x, \frac{x}{\varepsilon}\right). \tag{1.12}$$

Let $p\in\mathbb{R}$, *and let* $v=v(y,p)$ *be the unique solution in* $H^1_{per}(Y)$ *of the cell problem* (1.7)*. Define*

$$\bar{A}_i(p) := \frac{1}{|Y|}\int_Y A_i(y, v(y,p))\,dy. \tag{1.13}$$

Assume also that v_0 *is "well-prepared", i.e. satisfies*

$$v_0(x,y) = v(y, \bar{v}_0(x)) \tag{1.14}$$

for some $\bar{v}_0 \in L^1\cap L^\infty(\mathbb{R}^N)$.

Then as ε *goes to 0,*

$$v^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{v}(t,x)\right) \to 0 \quad \text{in } L^2_{loc}([0,\infty)\times\mathbb{R}^N),$$

where $\bar{v}=\bar{v}(t,x)\in\mathcal{C}([0,\infty), L^1(\mathbb{R}^N))\cap L^\infty([0,\infty)\times\mathbb{R}^N)$ *is the unique entropy solution of the hyperbolic scalar conservation law*

$$\begin{cases} \dfrac{\partial \bar{v}}{\partial t} + \displaystyle\sum_{i=1}^N \frac{\partial \bar{A}_i(\bar{v}(t,x))}{\partial x_i} = 0, \\ \bar{v}(t=0,x) = \bar{v}_0(x) \in L^1\cap L^\infty(\mathbb{R}^N). \end{cases} \tag{1.15}$$

Actually, the result proved in section 3 is more general than proposition 1.3, but is much more complicated to state at this stage. In particular, we work in a L^1 rather than L^∞ setting, which appears to us to be entirely new for this kind of equation; this point will be developed a little further in remark 3.2. We emphasize that inequality (1.10) was already used in [3], but we believe that the proof given here gives a better insight of the homogenization process.

Let us mention related results of Weinan E (see [6], [7]), and Weinan E and Denis Serre (see [8]), which use two-scale Young measures instead of the kinetic formulation in a hyperbolic context. In fact, the proof of [3] is close to the ones of these articles, although the viscous term in (1.11) is absent from the problems studied by Weinan E in [6], and Weinan E and Denis Serre in [8]. Indeed, the scaling in our problem is chosen so that the viscosity has the same order of magnitude than the size of the oscillations in the flux function, and thus the viscosity has an effect at a microscopic level only. Notice that the (macroscopic) homogenized problem (1.15) is hyperbolic; this justifies the use of hyperbolic tools, such as Young measures or a kinetic fomulation, in the study of equation (1.11).

We also wish to point out that the expression of the homogenized flux in the case studied by Weinan E and Denis Serre in [8] when $N = 1$ is the same as in (1.13). However, the corrector v appearing in the expression is not the same in both cases : indeed, in the hyperbolic case studied by Weinan E and Denis Serre, v is a solution of

$$\partial_y A(y, v(y,p)) = 0.$$

In particular, v is not unique in general, although the homogenized flux is. We refer the interested reader to [8] and [12] for details; the latter uses an equivalent formulation using Hamilton-Jacobi equations.

The organization of this article is as follows : first we derive a kinetic formulation for equation (1.1). As usual, this allows us to define a weaker notion of solutions of the parabolic conservation law (1.1), called *kinetic solutions*. We also derive formally the L^1 contraction principle for kinetic solutions of equation (1.1). Then we use this formulation to give another proof of proposition 1.3 in section 3. Eventually, in section 4 we give a rigorous proof for the derivation of the L^1 contraction principle announced in section 2.

2. Kinetic formulation . This section is devoted to the derivation of a kinetic formulation for equation (1.1). Throughout the section, we assume that the hypotheses of proposition 1.1 are satisfied, that is, $A \in W^{1,\infty}_{\text{per,loc}}(Y \times \mathbb{R})$, and A satisfies either (1.4), or (1.5), or (1.6). Additionnally, we assume that

$$a(y, \cdot) \in \mathcal{C}(\mathbb{R})^N \quad \text{for almost every } y \in Y. \tag{2.1}$$

Under such hypotheses, the following result is easily deduced from proposition 1.1 :

LEMMA 2.1. *For a.e.* $y \in Y$*,* $p \mapsto v(y,p)$ *is a* $\mathcal{C}^1$ *diffeomorphism from* $\mathbb{R}$ *to* $(\alpha_-(y), \alpha_+(y))$*, where* $\alpha_\pm(y) = \lim_{p\to\pm\infty} v(y,p)$*.*

Its reciprocal application is denoted by $w(y,\cdot)$

$$w(y) : (\alpha_-(y), \alpha_+(y)) \to \mathbb{R}.$$

REMARK 2.1. *Notice that* $+\infty$ *(resp.* $-\infty$*) is an admissible value for* α_+ *(resp.* α_-*). In fact, it can be checked that*

$$\langle \alpha_\pm \rangle = \pm\infty,$$

and there are cases when

$$\alpha_\pm(y) = \pm\infty \quad \forall y \in Y.$$

Indeed, for all $y \in Y$*, the family* $(v(y,p) - v(y,0))_{p>0}$ *is increasing in* p *and nonnegative. Moreover,*

$$\langle v(\cdot,p) - v(\cdot,0) \rangle = p \quad \forall p \in \mathbb{R}.$$

Hence according to Lebesgue's monotone convergence theorem, $\langle \alpha_+ - v(\cdot,0) \rangle = +\infty$*, and thus* $\langle \alpha_+ \rangle = +\infty$*. If we assume additionnally that* $m = 0$ *in hypothesis* (1.2) *(i.e. we assume that* (1.4) *is satisfied), then it is proved in [3], lemma 6, that*

$$\lim_{p\to+\infty} \inf_{y\in Y} v(y,p) = +\infty.$$

In that case, $\alpha_+(y) = +\infty$ *for all* $y \in Y$*.*

We begin our study of equation (1.1) with the following

LEMMA 2.2. *Let* $u \in \mathcal{C}([0,\infty); L^1(Y)) \cap L^2_{loc}(0,\infty; H^1_{per}(Y)) \cap L^\infty_{loc}([0,\infty) \times Y)$ *be an arbitrary solution of* (1.1)*. Assume that the flux* $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})$ *satisfies* (2.1) *and the hypotheses of proposition 1.1.*

Then the function u *satisfies the following equality in the sense of distributions on* $[0,\infty) \times Y \times \mathbb{R}_p$

$$\partial_t (u - v(y,p))_+ + \mathrm{div}_y \left[\mathbf{1}_{u>v(y,p)} (A(y,u) - A(y,v(y,p))) \right] - \Delta_y (u - v(y,p))_+ = -m, \tag{2.2}$$

where

$$m(t,y,p) = \frac{1}{\frac{\partial v}{\partial p}(y,p)} \left| \nabla_y (u(t,y) - v(y,p)) \right|^2 \delta(p = w(y,u(t,y)))$$

is a nonnegative measure on $(0,\infty) \times Y \times \mathbb{R}$*.*

Consequently, u *is an entropy solution of* (1.1) *in the sense of definition 1.2.*

We postpone the proof of lemma 2.2 to the end of section 2. Let us stress that equality (2.2) is to be understood in the sense of distributions in $[0,\infty) \times Y \times \mathbb{R}$. Such an equality would indeed be meaningless were it considered for $p \in \mathbb{R}$ fixed.

Let us now write down the kinetic formulation for equation (1.1). Let u be an entropy solution of (1.1); differentiating equality (2.2) with respect to p leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial v(y,p)}{\partial p} f^+ \right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v(y,p)}{\partial p} a_i(y, v(y,p)) f^+ \right) - \Delta_y \left(\frac{\partial v(y,p)}{\partial p} f^+ \right) = \frac{\partial m(t,y,p)}{\partial p} \tag{2.3}$$

where $f^+(t,y,p) = \mathbf{1}_{u(t,y)>v(y,p)}$.

The same kind of equation holds for $f^- = \mathbf{1}_{u(t,y)<v(y,p)} = 1 - f^+$ (recall (1.9))

$$\frac{\partial}{\partial t}\left(\frac{\partial v(y,p)}{\partial p} f^- \right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v(y,p)}{\partial p} a_i(y, v(y,p)) f^- \right) - \Delta_y \left(\frac{\partial v(y,p)}{\partial p} f^- \right) = -\frac{\partial m(t,y,p)}{\partial p} \tag{2.4}$$

This leads to a notion of kinetic solution :

Definition 2.3. *Assume that the flux A satisfies the hypotheses of proposition 1.1 and* (2.1)*. Let $u = u(t, y) \in \mathcal{C}([0, \infty); L^1(Y)) \cap L^2_{loc}(0, \infty; H^1_{per}(Y))$ such that*

$$\alpha_-(y) < u(t, y) < \alpha_+(y) \quad \text{for a.e. } (t, y) \in [0, \infty) \times Y.$$

We say that u is a kinetic solution of (1.1) *if $f^+ = \mathbf{1}_{u(t,y)>v(y,p)}$ satisfies* (2.3) *in the sense of distributions with the initial data $f^+(t = 0, y, p) = \mathbf{1}_{u_0(y)>v(y,p)}$, and if there exists a function $\mu \in L^\infty(\mathbb{R})$ such that $\mu(p) \to 0$ as $|p| \to \infty$, and*

$$\int_0^\infty \int_Y m(t, y, p)\, dy\, dt \leq \mu(p) \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{2.5}$$

Precisely, u is a kinetic solution of (1.1) *if* (2.5) *holds and if for all test function $\psi = \psi(t, y, p) \in \mathcal{D}_{per}([0, \infty) \times Y \times \mathbb{R})$, we have*

$$\begin{aligned}\int_0^\infty \int_{Y\times\mathbb{R}} f^+(t, y, p) \frac{\partial v(y, p)}{\partial p} \{\partial_t\psi + a_i(y, v(y, p))\partial_{y_i}\psi + \Delta_y\psi\}\, dt\, dy\, dp = \\ = \int_0^\infty \int_{Y\times\mathbb{R}} m(t, y, p)\partial_p\psi(t, y, p)\, dtdydp - \int_{Y\times\mathbb{R}} \mathbf{1}_{u_0(y)>v(y,p)} \frac{\partial v(y, p)}{\partial p}\psi(0, y, p)dydp.\end{aligned} \tag{2.6}$$

Notice that without any loss of generality, we can choose a function μ in (2.5) which is nonincreasing on $(0, \infty)$ and nondecreasing on $(-\infty, 0)$.

It is easily checked that the notions of entropy and kinetic solutions are equivalent as long as u is bounded in some kind of L^∞ norm :

Proposition 2.4. *Assume that A satisfies* (2.1) *and the hypotheses of proposition 1.1. Let $u = u(t, y) \in \mathcal{C}([0, \infty); L^1(Y)) \cap L^2_{loc}(0, \infty; H^1_{per}(Y))$. Assume that there exist real numbers $\beta_1, \beta_2 \in \mathbb{R}$ such that*

$$v(y, \beta_1) \leq u(t, y) \leq v(y, \beta_2) \quad \text{for a.e. } (t, y) \in (0, \infty) \times Y. \tag{2.7}$$

Then u is an entropy solution of (1.1) *if and only if u is a kinetic solution.*

We are then able to prove the L^1 contraction principle thanks to the kinetic formulation; we wish to emphasize that when u satisfies (2.7), this result is not new by any means, and has been known since the articles of Kruzkhov [10, 11]. However, we present here a different proof (see section 4), using merely regularizations by convolution following [16, 17]. Moreover, we prove that the L^1 contraction principle holds for a larger class of solutions.

Theorem 2.5. *Assume the hypotheses of proposition 1.1 are satisfied, with $a \in W^{1,1}_{per,loc}(Y \times \mathbb{R})^N$, and*

$$\partial_v a \in L^\infty_{loc}(Y \times \mathbb{R})^N, \tag{2.8}$$

$$\forall R > 0,\ \exists\, \alpha, C > 0, \forall (y, y') \in Y^2\ \forall v \in (-R, R) \quad |a(y, v) - a(y', v)| \leq C|y - y'|^\alpha. \tag{2.9}$$

Let u_1, u_2 be two kinetic solutions of (1.1)*. Then*

$$||(u_1(t) - u_2(t))_+||_{L^1(Y)} \leq ||(u_1(t = 0) - u_2(t = 0))_+||_{L^1(Y)}. \tag{2.10}$$

Moreover, if for all $T>0$

$$\int_0^T \int_Y \int_{\mathbb{R}} \frac{\partial v(y,p)}{\partial p} \left|a(y,v(y,p))\right| \mathbf{1}_{u_2(t,y)<v(y,p)<u_1(t,y)}\, dt\, dy\, dp < +\infty, \tag{2.11}$$

then the following inequality holds, in the sense of distributions on $[0,\infty)\times Y$

$$\frac{\partial}{\partial t}(u_1-u_2)_+ + \frac{\partial}{\partial y_i}\left[\mathbf{1}_{u_1>u_2}\left(A_i(y,u_1)-A_i(y,u_2)\right)\right] - \Delta_y (u_1-u_2)_+ \le 0. \tag{2.12}$$

Remark 2.2. *Hypothesis* (2.11) *is necessary in order to retrieve inequality* (2.12). *However, if the sole purpose is to derive the L^1 contraction inequality* (2.10), *hypothesis* (2.11) *is no longer required. Hypothesis* (2.11) *implies that the function*

$$(t,y)\mapsto \mathbf{1}_{u_1>u_2}\left[A(y,u_1(t,y)) - A(y,u_2(t,y))\right]$$

belongs to $L^1((0,T)\times Y)^N$ for all $T>0$. Notice that such an integrability property is not obvious in general, since we no longer assume that $u\in L^\infty_{loc}$, and thus $A(\cdot,u)$ does not belong to L^∞_{loc} either.

Let us explain formally how inequality (2.12) is derived : let u_1, u_2 be two kinetic solutions of (1.1). We set $f_1=\mathbf{1}_{u_1(t,y)>v(y,p)}$, $f_2=\mathbf{1}_{u_2(t,y)<v(y,p)}$,

$$m_i = |\nabla_y u_i(t,y) - \nabla_y v(y,p)|^2 \frac{1}{\frac{\partial v(y,p)}{\partial p}} \delta(p=w(y,u_i(t,y))), \quad i=1,2.$$

Then

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_1\right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v}{\partial p} a_i(y,v(y,p)) f_1\right) - \Delta_y\left(\frac{\partial v}{\partial p} f_1\right) = \frac{\partial m_1}{\partial p}, \tag{2.13}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_2\right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v}{\partial p} a_i(y,v(y,p)) f_2\right) - \Delta_y\left(\frac{\partial v}{\partial p} f_2\right) = -\frac{\partial m_2}{\partial p}. \tag{2.14}$$

Multiply (2.13) by f_2, and (2.14) by f_1; recalling equation (1.9), we add the two equations thus obtained and we are led to

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_1 f_2\right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v}{\partial p} a_i(y,v(y,p)) f_1 f_2\right) - \Delta_y\left(\frac{\partial v}{\partial p} f_1 f_2\right) = \\ = \frac{\partial m_1}{\partial p} f_2 - \frac{\partial m_2}{\partial p} f_1 - 2\frac{\partial v}{\partial p}\nabla_y f_1\cdot\nabla_y f_2.\end{aligned} \tag{2.15}$$

Set $\varphi_i(t,y)=w(y,u_i(t,y))$ $(i=1,2)$, i.e. $v(y,\varphi_i(t,y))=u_i(t,y)$. Then

$$\nabla_y \varphi_i(t,y) = \frac{1}{\frac{\partial v}{\partial p}(y,\varphi_i(t,y))}\left[\nabla_y u_i(t,y) - \nabla_y v(y,\varphi_i(t,y))\right].$$

Notice that

$$f_1 = \mathbf{1}_{u_1(t,y)>v(y,p)} = \mathbf{1}_{\varphi_1(t,y)>p},$$
$$f_2 = \mathbf{1}_{u_2(t,y)<v(y,p)} = \mathbf{1}_{\varphi_2(t,y)<p},$$

and thus, setting $\eta_1=1$ and $\eta_2=-1$,

$$\frac{\partial f_i}{\partial p} = -\eta_i\, \delta(p=\varphi_i(t,y)),$$
$$\nabla_y f_i = \eta_i \nabla_y \varphi_i(t,y)\delta(p=\varphi_1(t,y)).$$

We refer to the proof of lemma 2.2, at the end of the present section, for a derivation of the above equalities in the sense of distributions on $[0,\infty)\times Y\times\mathbb{R}$.

On the other hand, for any function $G\in W^{1,\infty}_{\text{loc}}(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} G'(v(y,p))f_1f_2\frac{\partial v(y,p)}{\partial p}\,dp &= \int_{\mathbb{R}} G'(v(y,p))\mathbf{1}_{u_2(t,y)<v(y,p)<u_1(t,y)}\frac{\partial v(y,p)}{\partial p}\,dp\\ &= \mathbf{1}_{u_2(t,y)<u_1(t,y)}\left[G(u_1(t,y))-G(u_2(t,y))\right].\end{aligned}$$

Hence, integrating (2.15) with respect to p on $\mathbb{R}$ yields

$$\begin{aligned}&\frac{\partial}{\partial t}(u_1-u_2)_+ + \frac{\partial}{\partial y_i}\mathbf{1}_{u_2(t,y)<u_1(t,y)}\left[A_i(y,u_1(t,y))-A_i(y,u_2(t,y))\right]-\Delta_y(u_1-u_2)_+\\ &=\int_{\mathbb{R}} -m_1\partial_p f_2+m_2\partial_p f_1-2\frac{\partial v}{\partial p}\nabla_y f_1\cdot\nabla_y f_2\,dp\\ &=-\int_{\mathbb{R}}|\nabla_y u_1(t,y)-\nabla_y v(y,\varphi_1)|^2\frac{1}{\frac{\partial v(y,p)}{\partial p}}\delta(p=\varphi_1)\delta(p=\varphi_2)dp\\ &\quad-\int_{\mathbb{R}}|\nabla_y u_2(t,y)-\nabla_y v(y,\varphi_2)|^2\frac{1}{\frac{\partial v(y,p)}{\partial p}}\delta(p=\varphi_2)\delta(p=\varphi_1)dp\\ &\quad+2\int_{\mathbb{R}}\frac{\partial v}{\partial p}(y,p)\,\nabla_y\varphi_1(t,y)\cdot\nabla_y\varphi_2(t,y)\delta(p=\varphi_1)\delta(p=\varphi_2)dp\\ &=-\int_{\mathbb{R}}\frac{1}{\frac{\partial v(y,p)}{\partial p}}\delta(p=\varphi_1)\delta(p=\varphi_2)|\nabla_y(u_1-u_2)(t,y)-\nabla_y v(y,\varphi_1)+\nabla_y v(y,\varphi_2)|^2\,dp\\ &\le 0\end{aligned}$$

which is exactly the L^1 contraction principle between u_1 and u_2.

However, the calculations above are entirely formal, since product of Dirac masses are not well-defined objects, and f_1,f_2 do not have enough regularity to perform nonlinear calculations. Thus, regularizations are necessary in order to justify the contraction principle, which is proved in section 4.

Proof of lemma 2.2. Notice first that since $u(t,y)$ and $v(y,p)$ are both solutions of (1.1), we always have

$$\partial_t\left[u(t,y)-v(y,p)\right]+\operatorname{div}_y\left[A(y,u)-A(y,v(y,p))\right]-\Delta_y\left[u(t,y)-v(y,p)\right]=0.$$

Thanks to the regularizing parabolic (resp. elliptic) term, the regularity of u (resp. v) is sufficient for us to use the chain rule, and thus

$$\begin{aligned}\mathbf{1}_{u(t,y)>v(y,p)}\partial_t\left[u(t,y)-v(y,p)\right]&=\partial_t\left[u(t,y)-v(y,p)\right]_+,\\ \mathbf{1}_{u(t,y)>v(y,p)}\operatorname{div}_y\left[A(y,u)-A(y,v(y,p))\right]&=\operatorname{div}_y\left[\mathbf{1}_{u(t,y)>v(y,p)}\left(A(y,u)-A(y,v(y,p))\right)\right],\\ \mathbf{1}_{u>v(y,p)}\Delta_y\left[u-v(y,p)\right]&=\Delta_y\left[u-v(y,p)\right]_+-\nabla_y\mathbf{1}_{u>v(y,p)}\cdot\nabla_y\left[u-v(y,p)\right]\end{aligned}$$

Similar calculations can be found for instance in [10, 11], and are in fact at the heart of Kruzkhov's method for proving the L^1 contraction principle.

The major difficulty comes from the term $\nabla_y\mathbf{1}_{u(t,y)>v(y,p)}$. Notice that $\mathbf{1}_{u(t,y)>v(y,p)}=\mathbf{1}_{w(y,u(t,y))>p}$. When $p\in\mathbb{R}$ is considered as a fixed parameter, we have

$$\nabla_y\mathbf{1}_{u>v(y,p)}=\nu\otimes\mathcal{H}^{n-1}_{\partial\omega}$$

where $\omega := \{y \in Y;\ w(y,u(t,y)) > p\}$, $\mathcal{H}^{n-1}_{\partial\omega}$ is the $(n-1)$-dimensional Hausdorff measure along $\{w(y,u(t,y)) = p\}$, and ν is the unit normal vector field oriented from $\{w(y,u(t,y)) < p\}$ to $\{w(y,u(t,y)) > p\}$. In general, no simplification occurs. However, when deriving a kinetic formulation for equation (1.1), we are only interested in the computation of $\nabla_y \mathbf{1}_{u>v(y,p)}$ in the sense of distributions on $[0,\infty) \times Y \times \mathbb{R}_p$ (see for instance [14], [13], and section 3.2 in [17]). In that case, we can give another expression for the gradient of $\mathbf{1}_{u>v(y,p)}$, namely

$$\begin{aligned}\nabla_y \mathbf{1}_{u(t,y)>v(y,p)} &= \nabla_y \mathbf{1}_{w(y,u(t,y))>p}\\ &= \nabla_y \left(w(y,u(t,y))\right) \delta(p = w(y,u(t,y)))\\ &= \frac{1}{\frac{\partial v}{\partial p}(y,p)} \nabla_y (u(t,y) - v(y,p)) \delta(p = w(y,u(t,y))).\end{aligned}$$

Notice that the above expression, although meaningless if considered for $p \in \mathbb{R}$ fixed, is well-defined in the sense of distributions on $[0,\infty) \times Y \times \mathbb{R}_p$.

Thus (2.2) is proved. Consequently, all solutions of (1.1) satisfy inequality (1.10) in the sense of distributions on $[0,\infty) \times Y \times \mathbb{R}$. And it is then easily checked that if a solution u of (1.1) satisfies (1.10) in the sense of distributions in t, y, p, then u satisfies (1.10) for all p in the sense of distributions in t, y. □

3. An application to homogenization. We provide here a proof for proposition 1.3. The kinetic formulation derived above allows a better understanding of the homogenization process, and the proof is much more elegant than the original one in [3], which used two-scale Young measures.

We will work in the context of kinetic solutions of equation (1.11) : let $\varepsilon > 0$, and let $u^\varepsilon \in L^\infty_{\text{loc}}([0,\infty); L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^2_{\text{loc}}(0,+\infty; H^1_{\text{loc}}(\mathbb{R}^N))$. We assume that

$$f^\varepsilon(t,x,p) := \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}$$

is a solution in the sense of distributions of

$$\begin{gathered}\partial_t \left(v_p\left(\frac{x}{\varepsilon},p\right) f^\varepsilon\right) + \partial_{x_i}\left[a_i\left(\frac{x}{\varepsilon}, v\left(\frac{x}{\varepsilon},p\right)\right) v_p\left(\frac{x}{\varepsilon},p\right) f^\varepsilon\right] - \varepsilon\Delta_x\left(v_p\left(\frac{x}{\varepsilon},p\right) f^\varepsilon\right) = \partial_p m^\varepsilon,\\ f^\varepsilon(t=0) = \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u_0\left(x,\frac{x}{\varepsilon}\right)}\end{gathered} \tag{3.1}$$

where

$$m^\varepsilon(t,x,p) := \varepsilon\left|\nabla_x u^\varepsilon(t,x) - \nabla_y v\left(\frac{x}{\varepsilon},p\right)\right|^2 \frac{1}{v_p\left(\frac{x}{\varepsilon},p\right)} \delta\left(p = w\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right)\right).$$

We assume that the hypotheses of proposition 1.1 are satisfied, together with (2.1), so that $w(y,p)$ is well-defined (see lemma 2.1). We have used the notation $v_p(y,p) = \partial_p v(y,p)$.

The hypotheses on f^ε are the following:

(H1) $u_0(x,y) = v(y,\bar{u}_0(x))$, for some $\bar{u}_0 \in L^1(\mathbb{R}^N)$;

(H2) $u_0 - v(y,0) \in L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$; this means that

$$\int_{\mathbb{R}^N} \sup_{y\in Y} |v(y,\bar{u}_0(x)) - v(y,0)|\,dx < +\infty,$$

which is slightly stronger than requiring $\bar{u}_0 \in L^1$.

(H3) $f^\varepsilon(t,x,p) \to 0$ (resp. $1 - f^\varepsilon \to 0$) as $p \to +\infty$ (resp. as $p \to -\infty$) for a.e. $(t,x) \in [0,\infty) \times \mathbb{R}^N$ and for all $\varepsilon > 0$. Equivalently,

$$\alpha_-\left(\frac{x}{\varepsilon}\right) < u^\varepsilon(t,x) < \alpha_+\left(\frac{x}{\varepsilon}\right) \quad \text{for a.e. } (t,x) \in (0,\infty) \times \mathbb{R}^N,$$

where α_- and α_+ were defined in lemma 2.1.

(H4) For all $\varepsilon > 0$, there exists a function $\mu_\varepsilon \in L^\infty(\mathbb{R})$ such that $\mu_\varepsilon(p) \to 0$ as $|p| \to \infty$ and

$$\int_0^\infty \int_{\mathbb{R}^N} m^\varepsilon(t,x,p)\, dt\, dx \leq \mu_\varepsilon(p) \quad \forall p \in \mathbb{R}.$$

(H5) For all $\varepsilon > 0$, the function

$$(t,x,p) \mapsto \frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon}, p\right)\left[\mathbf{1}_{p>0} f^\varepsilon(t,x,p) + \mathbf{1}_{p<0}(1 - f^\varepsilon(t,x,p))\right]$$

belongs to $L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R}^{N+1}))$. Equivalently, the function

$$(t,x) \mapsto u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, 0\right)$$

belongs to $L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R}^N))$.

A function $u^\varepsilon \in L^\infty_{\text{loc}}([0,\infty); L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^2_{\text{loc}}(0,+\infty; H^1_{\text{loc}}(\mathbb{R}^N))$ such that f^ε is a solution of (3.1) and such that **(H3)-(H5)** are satisfied is called a *kinetic solution* of the parabolic scalar conservation law (1.11). Notice that we do not assume that (1.11) is satisfied in the sense of distributions.

Let us now state the result we prove in this section.

THEOREM 3.1. *Assume that A satisfies the hypotheses of proposition 1.1 and* (2.1). *Let* $u^\varepsilon \in L^\infty_{loc}([0,\infty); L^1_{loc}(\mathbb{R}^N)) \cap L^2_{loc}(0,+\infty; H^1_{loc}(\mathbb{R}^N))$ *be a kinetic solution of* (1.11) *such that hypotheses* **(H1) - (H5)** *are satisfied. Then*

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0$$

in $L^1_{loc}([0,\infty) \times \mathbb{R}^N)$, *where* $\bar{u} \in L^\infty([0,\infty), L^1(\mathbb{R}^N))$ *is the kinetic solution of* (1.15) *with initial data* $\bar{u}_0$.

REMARK 3.1. *When hypothesis* **(H1)** *on the microscopic profile of the initial data is not satisfied, it is proved in the* L^∞ *case in [4] that there is an initial layer of typical size* ε*, during which the solution adapts itself to the profile dictated by the microscopic structure. The proof of this result relies on the parabolic structure of the equation, which cannot be used here since the kinetic formulation is essentially a hyperbolic tool.*

REMARK 3.2. *It is can be checked that* **(H2) - (H5)** *are always satisfied when* $\bar{u}_0 \in L^\infty \cap L^1(\mathbb{R}^N)$ *and* $u^\varepsilon \in L^\infty_{loc}$ *is an entropy solution. However, we wish to stress that hypothesis* **(H3)** *does not imply that* $u^\varepsilon \in L^\infty_{loc}([0,\infty) \times \mathbb{R}^N)$ *in general. For instance, in the case when hypothesis* (1.4) *is satisfied, we have* $\alpha_\pm = \pm\infty$*, as explained in remark 2.1. Hence in that case, hypothesis* **(H3)** *is always satisfied, and the only bound required on* u^ε *is* **(H5)**, *which is an* L^1 *bound. Consequently, we refer to* **(H2) - (H5)** *as an "*L^1 *setting", by contrast with the "*L^∞ *setting" of entropy solutions.*

At last, let us mention that the function μ_ε in hypothesis **(H4)** *can in fact be derived from equation* (3.1) *(see lemma 3.2 below), if it is known that* **(H4)** *holds for some function μ_ε; nonetheless,* **(H4)** *cannot be avoided and is necessary for lemma 3.2 to hold.*

We will prove the convergence in several steps; first, we introduce the two-scale weak limit $f(t,x,y,p)$ of f^ε. Then, the key point in the analysis is to show that $f(t,x,y,p) = \mathbf{1}_{p<\bar{u}(t,x)}$, where $\bar{u}$ is the solution of the homogenized problem. Hence, we first prove that f does not depend on y. Then we derive the macroscopic equation solved by f and we prove that $f(t=0) = \mathbf{1}_{p<\bar{u}_0(x)}$; this entails that $f = \mathbf{1}_{p<\bar{u}}$, and $\bar{u}$ can be identified thanks to the equation solved by f. Eventually, we prove the strong convergence in L^1_{loc}.

We begin with a few preliminary bounds on m^ε and f^ε, of which we only give a rough idea of the proof (see for instance [17], proposition 4.1.7 and lemma 3.1.7 for the derivation of similar inequalities):

Lemma 3.2. *Assume that* **(H1) - (H5)** *are satisfied.*

- *There exists a constant $C > 0$ such that for all $\varepsilon > 0$, for a.e. $t > 0$,*

$$\int_{\mathbb{R}^{N+1}} v_p\left(\frac{x}{\varepsilon}, p\right)\left(\mathbf{1}_{p>0} f^\varepsilon(t,x,p) + \mathbf{1}_{p<0}(1-f^\varepsilon)(t,x,p)\right)\,dx\,dp \le C.$$

- *There exists a constant $C > 0$ such that for all $p_0 > 0$, $\varepsilon > 0$,*

$$\int_0^\infty \int_{\mathbb{R}^N} m^\varepsilon(t,x,p_0)\,dx\,dt \le \int_{\mathbb{R}^N} \left(v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right)_+ dx \le C.$$

  The same kind of bound holds for $p_0 < 0$.
  Thus $m^\varepsilon((0,+\infty)\times\mathbb{R}^N\times(-R,R))$ is bounded for all $R > 0$ uniformly in ε.
- *For all $t \ge 0$, for all $p_0 > 0$, for all $\varepsilon > 0$*

$$\int_{\mathbb{R}^N} \left(u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, p_0\right)\right)_+ dx \le \int_{\mathbb{R}^N} \left(v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right)_+ dx \tag{3.2}$$

We deduce from the second bound in the lemma that we can take in **(H4)**

$$\begin{aligned}\mu_\varepsilon(p) := \mathbf{1}_{p>0} &\int_{\mathbb{R}^N} \left(v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p\right)\right)_+ dx \\ &+ \mathbf{1}_{p<0} \int_{\mathbb{R}^N} \left(v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p\right)\right)_- dx.\end{aligned}$$

Then μ_ε is bounded in L^∞, uniformly in ε. Moreover, it will be proved in the very last step of the proof that for all p, $\mu_\varepsilon(p)$ converges as $\varepsilon \to 0$ towards $\mu_0(p)$, for some function $\mu_0 \in L^\infty(\mathbb{R})$ vanishing at infinity.

Proof. Thanks to the integrability assumptions **(H4)-(H5)** on f^ε and m^ε, we prove that for any test function $S' \in \mathcal{D}(\mathbb{R})$, for all $t > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} S'(p) f^\varepsilon(t,x,p) v_p\left(\frac{x}{\varepsilon}, p\right) dx\,dp - \int_{\mathbb{R}^{N+1}} S'(p) f^\varepsilon(t=0,x,p) v_p\left(\frac{x}{\varepsilon}, p\right) dx\,dp = \\ = -\int_0^t \int_{\mathbb{R}^{N+1}} m^\varepsilon(t,x,p) S''(p)\,dt\,dx\,dp.\end{aligned}$$

Then, using the fact that μ_ε vanishes at infinity , we prove that the above equality holds for more general functions S. In particular, the choice $S'(p) = \mathbf{1}_{p>0}$ (and thus

$S''(p) = \delta(p=0)$) yields the first bound on f^ε, and the choice $S'(p) = \mathbf{1}_{p>p_0}$ for some $p_0 > 0$ gives the one on m^ε. Moreover

$$\int_{\mathbb{R}^{N+1}} \mathbf{1}_{p>p_0} f^\varepsilon(t,x,p) v_p\left(\frac{x}{\varepsilon}, p\right)\, dx\, dp = \int_{\mathbb{R}^{N+1}} \mathbf{1}_{v\left(\frac{x}{\varepsilon},p_0\right)<v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)} v_p\left(\frac{x}{\varepsilon}, p\right)\, dx\, dp$$
$$= \int_{\mathbb{R}^N} \left[u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+ dx,$$

and thus the choice $S'(p) = \mathbf{1}_{p>p_0}$ also yields the bound on u^ε. □

We now use the concept of two scale convergence, defined by Grégoire Allaire in [1] following an idea of Gabriel N'Guetseng (see [15]), in order to find a two-scale limit for f^ε :

Proposition 3.3. *Let $\{v^\varepsilon\}_{\varepsilon>0}$ be a bounded sequence of $L^2(\Omega)$, where Ω is an open set of $\mathbb{R}^N$. Then as $\varepsilon \to 0$, there exists a subsequence, still denoted by ε, and $v^0 \in L^2(\Omega \times Y)$, such that*

$$\int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right) v^\varepsilon(x)\, dx \to \int_{\Omega\times Y} \psi(x,y) v^0(x,y)\, dx\, dy$$

for all $\psi \in \mathcal{C}_{per}(Y, L^2(\Omega))$.

It is then said that the sequence $\{v^\varepsilon\}_{\varepsilon>0}$ "two-scale" converges to v_0.

This concept is easily generalized to functions in L^∞ (the proof goes along the same lines as the one given in [1]), which allows us to prove the following :

Lemma 3.4. *There exists a function $f(t,x,y,p) \in L^\infty((0,\infty)\times\mathbb{R}^N \times Y \times \mathbb{R})$ and a subsequence, still denoted by ε, such that f^ε two-scale converges to f.*

It is easily checked that $0 \le f \le 1$ a.e. Since v_p, f and $1-f$ are nonnegative, lemma 3.2 entails that there exists a constant C such that

$$\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left\{\mathbf{1}_{p>0} f(t,x,y,p) + \mathbf{1}_{p<0}(1 - f(t,x,y,p))\right\} v_p(y,p)\, dx\, dy\, dp \le C \quad \text{a.e. t>0.}$$

The goal is now to identify the equations solved by f in order to prove that f is an indicator function. Hence, we now focus on the microscopic (i.e. in y) and macroscopic (i.e. in t,x) equations solved by f.

First step. Microscopic profile. Multiplying (3.1) by a test function of the form $\varepsilon\varphi(t,x,x/\varepsilon,p)$, with $\varphi \in \mathcal{D}_{\rm per}((0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$ and passing to the limit as $\varepsilon \to 0$ leads to the equation

$$-\Delta_y\left(\frac{\partial v}{\partial p} f\right) + \mathrm{div}_y\left(a(y, v(y,p))\frac{\partial v}{\partial p} f\right) = 0 \tag{3.3}$$

in the sense of distributions on $(0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R}$. Let us point out that $a(y, v(y,p))$ is an "admissible" test function in the sense of G. Allaire (see [1]) thanks to the continuity assumption (2.1).

Then, we regularize the equation (3.3) in the variables t,x,y,p thanks to a convolution kernel, and pass to the limit as the parameter of the regularization vanishes. We easily deduce that equation (3.3) in fact holds almost everywhere in t,x,p, in the variational sense in y.

Notice that the constant function equal to 1 on Y, denoted by $\bar{1}$, is a positive solution of the dual problem

$$-\Delta_y \bar{1} - a(y, v(y,p))\cdot\nabla_y \bar{1} = 0.$$

Consequently, by the Krein-Rutman theorem, we infer that any solution g of the equation

$$-\Delta_y g + \mathrm{div}_y \left(a(y, v(y,p))g\right) = 0$$

can be written $g(y) = c\frac{\partial v(y,p)}{\partial p}$, where c is a constant in y.

Thus $f(t,x,y,p)$ does not depend on y, and $f = f(t,x,p)$.

Second step. Evolution equation. Now, we multiply (3.1) by a test function of the form $\varphi(t,x,p)$, with $\varphi(t,x,p) = 0$ when $|p| \geq R$, $R > 0$ arbitrary; thanks to lemma 3.2, $m^\varepsilon((0,\infty)\times\mathbb{R}^N\times(-R,R))$ is bounded uniformly in ε, and thus up to the extraction of a subsequence, there exists a measure $\bar{m}_R$ such that

$$m^\varepsilon \rightharpoonup \bar{m}_R \quad \text{in } w - M^1((0,\infty)\times\mathbb{R}^N\times(-R,R)).$$

We define, for any $p\in\mathbb{R}$,

$$\bar{a}(p) = \frac{1}{|Y|}\int_Y a(y, v(y,p))\frac{\partial v}{\partial p}\,dy;$$

recall also that

$$\frac{1}{|Y|}\int_Y \frac{\partial v}{\partial p}\,dy = 1.$$

Then f satisfies, in the sense of distributions on $(0,\infty)\times\mathbb{R}^N\times(-R,R)$

$$\partial_t f + \mathrm{div}_x(\bar{a}(p)f) = \frac{\partial \bar{m}_R}{\partial p}. \tag{3.4}$$

We deduce that for any $0 < R < R'$, $\bar{m}_R = \bar{m}_{R'}$ on $(0,\infty)\times\mathbb{R}^N\times(-R,R)$. Consequently, the measure $\bar{m}$, defined by $\bar{m} = \bar{m}_R$ on $(0,\infty)\times\mathbb{R}^N\times(-R,R)$ is well-defined. Hence equation (3.4) holds in $(0,\infty)\times\mathbb{R}^{N+1}$ with $\bar{m}_R$ replaced by $\bar{m}$, and $\bar{m}\in\mathcal{C}(\mathbb{R}_p, w-M^1([0,\infty)\times\mathbb{R}^N_x)$. Moreover the measure $\bar{m}$ inherits from the bounds on m^ε : in particular, for almost every $p\in\mathbb{R}$,

$$\int_0^\infty\int_{\mathbb{R}^N\times Y}\bar{m}(t,x,y,p)\,dt\,dx\,dy \leq \mu_0(p), \tag{3.5}$$

and μ_0 belongs to L^∞ and vanishes at infinity.

Equation (3.4) looks very much like the kinetic formulation for a homogeneous and hyperbolic scalar conservation law (see for instance [14], [13], and [17], chapter 3). However we have to work out a few points before jumping to a conclusion.

Third step. Identification of f as an indicator function. First, the function which occurs in the kinetic formulation is the function $\chi:\mathbb{R}^2\to\{1,-1,0\}$ defined by

$$\chi(v,u) := \begin{cases} 1 & \text{if } 0 < v < u,\\ -1 & \text{if } u < v < 0,\\ 0 & \text{otherwise.}\end{cases}$$

Here, if $u^\varepsilon(t,x) - v\left(x/\varepsilon, \bar{u}(t,x)\right)$ converges strongly to 0, as we intend to prove, then $f = \mathbf{1}_{v(y,p)<v(y,\bar{u}(t,x))} = \mathbf{1}_{p<\bar{u}(t,x)}$; hence, a good candidate for a function $\chi(v,\bar{u}(t,x))$ seems to be

$$g(t,x,p) = \mathbf{1}_{p>0}f - \mathbf{1}_{p<0}(1-f) = f - \mathbf{1}_{p<0}.$$

The function g satisfies the same equation as f, and

$$\operatorname{sgn}(p)g = \mathbf{1}_{p>0}f + \mathbf{1}_{p<0}(1-f) = |g| \in [0,1].$$

Moreover,

$$\frac{\partial g}{\partial p} = \delta(p=0) + \partial_p f. \tag{3.6}$$

Recall that

$$\partial_p f^\varepsilon(t,x,p) = -\delta\left(p - w\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right)\right);$$

Hence $-\partial_p f^\varepsilon(t,x,p)$ is a nonnegative measure, uniformly bounded in ε on compact sets of $(0,\infty)\times\mathbb{R}^{N+1}$. Since $\partial_p f^\varepsilon$ weakly converges to $\partial_p f$, we deduce that $\partial_p f$ is a nonpositive locally finite measure.

There remains to check that

$$g(t=0,x,p) = \chi(p, \bar{u}_0(x)); \tag{3.7}$$

this equality is in fact not obvious : if $f^\varepsilon(t=0,x,p) = f_0\left(x, x/\varepsilon, p\right)$, then it is false in general that $f(t=0,x,y,p) = f_0(x,y,p)$. Indeed, there might be initial layers of typical size ε. These are not taken into account when passing to the two-scale limit because the test functions do not select the microscopic information in time. In order to see the possible initial layers, we should have taken test functions of the kind

$$\psi\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}, p\right).$$

Here, it is unnecessary to consider test functions which have microscopic oscillations in time because the initial data is well-prepared. Hence, there is no initial layer in this case. In other words, the u^ε are uniformly continuous in time at time $t=0$ (with values in L^1_{loc}). In terms of the kinetic formulation, this result follows directly from the fact that

$$f^\varepsilon(t=0,x,p) = \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right) < v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)} = \mathbf{1}_{p<\bar{u}_0(x)}.$$

Hence $f^\varepsilon(t=0)$ does not depend on ε. Consequently, multiplying (3.1) by a test function $\varphi(t,x,p) \in \mathcal{D}([0,\infty)\times\mathbb{R}^{N+1})$ yields

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^{N+1}} f^\varepsilon(t,x,p) \frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon},p\right)\left\{\partial_t\varphi + a_i\left(\frac{x}{\varepsilon}, v\left(\frac{x}{\varepsilon},p\right)\right)\partial_{x_i}\varphi + \varepsilon\Delta_x\varphi\right\}\,dt\,dx\,dp \\ = &\int_0^\infty \int_{\mathbb{R}^{N+1}} m^\varepsilon(t,x,p)\partial_p\varphi(t,x,p)dtdxdp - \int_{\mathbb{R}^{N+1}} \frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon},p\right)\mathbf{1}_{p<\bar{u}_0(x)}\varphi(t=0,x,p)dxdp\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ entails that

$$f(t=0,x,p) = \mathbf{1}_{p<\bar{u}_0(x)},$$

and thus

$$g(t=0,x,p) = \chi(p, \bar{u}_0(x)).$$

Gathering (3.4), (3.5), (3.6), (3.7), we infer that g is a generalized kinetic solution (see definition 4.1.2 in [17]) of the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial \bar{A}_i(u)}{\partial x_i} = 0,$$

where

$$\bar{A}_i'(p) = \bar{a}_i(p).$$

Now, we can apply theorem 4.3.1 in [17] : there exists $\bar{u}(t,x) \in L^\infty([0,\infty); L^1(\mathbb{R}^N))$ such that $g(t,x,p) = \chi(p, \bar{u}(t,x))$ a.e., and $\bar{u}$ is a kinetic solution of the above scalar conservation law.

And since

$$\mathbf{1}_{p>0} f - \mathbf{1}_{p<0}(1-f) = \chi(p, \bar{u}(t,x)),$$

we deduce that

$$f(t,x,p) = \mathbf{1}_{p<\bar{u}(t,x)}$$

almost everywhere.

Fourth step. Strong convergence. Let us now prove that this result entails that

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0$$

in L^1_{loc}.

(i) *Convergence of $u^\varepsilon \wedge v(x/\varepsilon, p_0)$ for all $p_0 > 0$:* take an arbitrary cut-off function $\varphi = \varphi(t,x)$ with compact support in $[0,\infty) \times \mathbb{R}^N$, $p_0 > 0$ and set

$$\psi(t,x,y,p) := \mathbf{1}_{\bar{u}(t,x)<p<p_0} \frac{\partial v}{\partial p}(y,p)\, \varphi(t,x).$$

Since $f^\varepsilon(t,x,p) = \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}$ two-scale converges to $f = \mathbf{1}_{p<\bar{u}(t,x)}$, we deduce that as $\varepsilon \to 0$

$$\int_0^\infty \int_{\mathbb{R}^{N+1}} \psi\left(t,x,\frac{x}{\varepsilon},p\right) \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}\, dp\, dx\, dt \to 0.$$

And the left-hand side can be transformed as follows

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^{N+1}} \psi\left(t,x,\frac{x}{\varepsilon},p\right) \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}\, dp\, dx\, dt \\
&= \int_0^\infty \int_{\mathbb{R}^{N+1}} \mathbf{1}_{v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)<v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)\wedge v\left(\frac{x}{\varepsilon},p_0\right)} \varphi(t,x) \frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon},p\right) dp\, dx\, dt \\
&= \int_0^\infty \int_{\mathbb{R}^{N+1}} \mathbf{1}_{v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)<v<u^\varepsilon(t,x)\wedge v\left(\frac{x}{\varepsilon},p_0\right)} \varphi(t,x) dv\, dx\, dt \\
&= \int_0^\infty \int_{\mathbb{R}^N} \varphi(t,x) \left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon},p_0\right) - v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)\right]_+ dx\, dt.
\end{aligned}$$

Take any compact set $K \subset [0,\infty)\times\mathbb{R}^N$, and choose a test function $\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R}^N)$ such that $0 \le \varphi \le 1$, and $\varphi \equiv 1$ on K. Then for all $\varepsilon > 0$,

$$\begin{aligned}
&\left|\left|\left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon},p_0\right) - v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(K)} \\
&\le \int_0^\infty \int_{\mathbb{R}^N} \varphi(t,x) \left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon},p_0\right) - v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)\right]_+ dx\, dt
\end{aligned}$$

In the inequality above, we have used the fact that $u_+ = \max(u,0)$ is always nonnegative. Thus we deduce that for all $p_0 > 0$

$$\left|\left|\left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^N)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The same kind of result holds for $p_0 < 0$.

(ii) *Convergence of u^ε:* let $T > 0$, $R > 0$, and set $Q := (0,T) \times B_R$. For $p_0 > 0$ arbitrary, we have

$$\begin{aligned}
&\left|\left|\left[u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(Q)} \\
\le &\left|\left|\left[u^\varepsilon \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(Q)} + \left|\left|\left[u^\varepsilon - u^\varepsilon \wedge v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+\right|\right|_{L^1(Q)} \\
\le &\left|\left|\left[u^\varepsilon \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(Q)} + \left|\left|\left[u^\varepsilon - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+\right|\right|_{L^1(Q)} \\
\le &\left|\left|\left[u^\varepsilon \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(Q)} + T\int_{\mathbb{R}^N}\left[v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+ dx
\end{aligned}$$

thanks to inequality (3.2).

According to **(H2)**, we have $[v(y,\bar{u}_0) - v(y,p_0)]_+ \in L^1(\mathbb{R}^N; \mathcal{C}_{\mathrm{per}}(Y))$; thus, using to a result of Grégoire Allaire (see [1]), we deduce

$$\int_{\mathbb{R}^N}\left[v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+ dx \to \int_{\mathbb{R}^N\times Y}\left[v\left(y, \bar{u}_0(x)\right) - v\left(y, p_0\right)\right]_+ dx\, dy$$

as $\varepsilon \to 0$, for all $p_0 > 0$. Since $||\,(v(y,p) - v(y,p'))_+\,||_{L^1(Y)} = (p-p')_+$ for all $p, p' \in \mathbb{R}$, we derive the following bound

$$\begin{aligned}
&\int_0^T\int_{B_R}\left[u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+ dx\, dt \\
\le &\int_0^T\int_{B_R}\left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+ dx\, dt \\
&+T\left|\int_{\mathbb{R}^N}\left[v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+ dx - ||(\bar{u}_0 - p_0)_+||_{L^1(\mathbb{R}^N)}\right| \\
&+T||(\bar{u}_0 - p_0)_+||_{L^1(\mathbb{R}^N)}
\end{aligned}$$

In the above inequality, take p_0 large enough so that $||(\bar{u}_0 - p_0)_+||_{L^1(\mathbb{R}^N)}$ is small enough, and then for this p_0, take $\varepsilon > 0$ small enough so that the two other terms are small (notice that the first one vanishes thanks to the first step). We deduce that

$$\left[u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+ \to 0$$

in $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^N)$, and theorem 3.1 is proved.

Moreover, we have proved that for all $p > 0$,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\mu_\varepsilon(p) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left[v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p\right)\right]_+ dx \\
&= \int_{\mathbb{R}^N\times Y}\left[v\left(y, \bar{u}_0(x)\right) - v\left(y, p\right)\right]_+ dx\, dy \\
&= ||(\bar{u}_0 - p)_+||_{L^1(\mathbb{R}^N)} =: \mu_0(p).
\end{aligned}$$

Thus μ_0 vanishes at infinity, and the result stated after lemma 3.2 holds.

4. Rigorous proof of the L^1 contraction principle. This section is devoted to the proof of inequality (2.12) under assumption (2.11) and the hypotheses of theorem 2.5. The main ideas behind the proof were exposed in the formal calculations in section 2; however, regularizations are necessary in order to justify nonlinear manipulations of the type

$$f_1\partial_t f_2 + f_2\partial_t f_1 = \partial_t(f_1 f_2),$$

as well as the reduction of the right hand-side.

As in [16], [17] (Chapter 4), we will merely regularize the equation by convolution; let $\varepsilon > 0$ be a small parameter, $\zeta_1 \in \mathcal{D}(\mathbb{R})$, $\zeta_2 \in \mathcal{D}(\mathbb{R}^N)$, $\zeta_3 \in \mathcal{D}(\mathbb{R})$ such that

$$\zeta_i \geq 0 \quad (i = 1, 2, 3),$$
$$\text{supp } \zeta_1 \subset [-1, 0], \quad \text{supp } \zeta_2 \subset B_1, \quad \text{supp } \zeta_3 \subset [-1, 1], \quad \zeta_1(0) = 0,$$
$$\int_{\mathbb{R}} \zeta_1 = \int_{\mathbb{R}^N} \zeta_2 = \int_{\mathbb{R}} \zeta_3 = 1.$$

We set, for $\varepsilon > 0$, $(t, x, p) \in \mathbb{R}^{N+2}$

$$\phi_\varepsilon(t, x, p) := \frac{1}{\varepsilon^{N+2}}\zeta_1\left(\frac{t}{\varepsilon}\right)\zeta_2\left(\frac{x}{\varepsilon}\right)\zeta_3\left(\frac{p}{\varepsilon}\right),$$

and for $(t, x, p) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}$

$$f_i^\varepsilon(t, x, p) = \int_{\mathbb{R}^{N+2}} f_i(s, z, q)\phi_\varepsilon(t - s, x - z, p - q)\, ds\, dz\, dq,$$
$$m_i^\varepsilon(t, x, p) = \int_{\mathbb{R}^{N+2}} m_i(s, z, q)\phi_\varepsilon(t - s, x - z, p - q)\, ds\, dz\, dq.$$

(Notice that the convolution in the space variable x is meant in the whole of $\mathbb{R}^N$: f_i is thus considered as a function defined on $[0, \infty) \times \mathbb{R}^N \times \mathbb{R}$, periodic with period Y in its second variable. The function f_i^ε is of course Y-periodic as well.)

We begin with the derivation of the equation solved by f^ε :

LEMMA 4.1. *Set $\tilde{a}_i(y, p) = a_i(y, v(y, p))\frac{\partial v(y,p)}{\partial p}$ for $1 \leq i \leq N$, $y \in Y$, $p \in \mathbb{R}$. Then for $\varepsilon < 1/2$, f_j^ε $(j = 1, 2)$ is a classical solution of*

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_j^\varepsilon\right) + \frac{\partial}{\partial y_i}\left(\tilde{a}_i(y, p) f_j^\varepsilon\right) - \Delta_y\left(\frac{\partial v}{\partial p} f_j^\varepsilon\right) = \eta_j \frac{\partial m_j^\varepsilon}{\partial p} + r_j^\varepsilon \tag{4.1}$$

where $\eta_1 = 1$, $\eta_2 = -1$, and the error term r_j^ε is equal to

$$\begin{aligned} r_j^\varepsilon(t, y, p) = {} & \frac{\partial}{\partial t}\left[\frac{\partial v}{\partial p}(y, p) f_j^\varepsilon(t, y, p) - \left(\frac{\partial v}{\partial p} f_j\right) * \phi_\varepsilon(t, y, p)\right] \\ & + \frac{\partial}{\partial y_i}\left[\tilde{a}_i(y, p) f_j^\varepsilon(t, y, p) - (\tilde{a}_i f_j) * \phi_\varepsilon(t, y, p)\right] \\ & - \Delta_y\left[\frac{\partial v}{\partial p}(y, p) f_j^\varepsilon(t, y, p) - \left(\frac{\partial v}{\partial p} f_j\right) * \phi_\varepsilon(t, y, p)\right]. \end{aligned}$$

Moreover, for all $0 < \varepsilon < 1/2$, for all $p \in \mathbb{R}$,

$$\int_0^\infty \int_Y m_i^\varepsilon(t, y, p)\, dt\, dy \leq \max(\mu_i(p + 1), \mu_i(p - 1)),$$

where the functions μ_i were introduced in hypothesis (2.5) *in definition 2.3.*

We postpone the proof of lemma 4.1 to the end of the section.

Now, since f_j^ε is smooth we can multiply (4.1) written for f_1^ε (resp. f_2^ε) by f_2^ε (resp. f_1^ε), and add the two equations thus obtained. Following the calculations in section 2 leads to

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p}f_1^\varepsilon f_2^\varepsilon\right)+\frac{\partial}{\partial y_i}\left(\tilde a_i(y,p)f_1^\varepsilon f_2^\varepsilon\right)-\Delta_y\left(\frac{\partial v}{\partial p}f_1^\varepsilon f_2^\varepsilon\right)\\ =&\frac{\partial m_1^\varepsilon}{\partial p}f_2^\varepsilon-\frac{\partial m_2^\varepsilon}{\partial p}f_1^\varepsilon-2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon\cdot\nabla_y f_2^\varepsilon\\ &+r_1^\varepsilon f_2^\varepsilon+r_2^\varepsilon f_1^\varepsilon.\end{aligned}$$

Let $R>0$ arbitrary, and let $K_R\in\mathcal{D}(\mathbb{R})$ be a cut-off function such that

$$\begin{gathered}0\le K_R(p)\le 1,\quad |K_R'(p)|\le 2\quad\forall p\in\mathbb{R},\\ K_R(p)=1\quad\forall p\in[-R,R],\\ K_R(p)=0\quad\forall p\in(-\infty,-R-1]\cup[R+1,+\infty).\end{gathered}$$

Classically, the following convergence results hold for any test function $\theta=\theta(t,y)\in\mathcal{D}_{\text{per}}([0,\infty)\times Y)$ (recall (2.11))

$$\begin{aligned}&\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_0^\infty\int_{Y\times\mathbb{R}}\frac{\partial v}{\partial p}(y,p)f_1^\varepsilon f_2^\varepsilon\theta(t,y)K_R(p)\,dt\,dy\,dp=\int_Y(u_1-u_2)_+\theta(t,y)\,dt\,dy,\\ &\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_0^\infty\int_{Y\times\mathbb{R}}\tilde a_i(y,p)f_1^\varepsilon f_2^\varepsilon\partial_{y_i}\theta(t,y)K_R(p)\,dt\,dy\,dp=\\ &\qquad\qquad=\int_0^\infty\int_Y\mathbf{1}_{u_1>u_2}\left[A_i(y,u_1)-A_i(y,u_2)\right]\partial_{y_i}\theta(t,y)\,dt\,dy.\end{aligned}$$

(If one is interested in deriving (2.10), without assumption (2.11), instead of (2.12), one should merely take $\theta\in\mathcal{D}([0,\infty))$, independent of y, at this stage; the left-hand side in the second equality above is zero in that case. The rest of the proof remains unchanged.)

On the other hand, it is easily proved that the first order terms in r_j^ε go to 0 in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^{N+1})$ as $\varepsilon\to 0$ thanks to the assumption $a\in W^{1,1}_{\text{loc}}$. Hence, we now focus on the convergence of

$$\frac{\partial m_1^\varepsilon}{\partial p}f_2^\varepsilon-\frac{\partial m_2^\varepsilon}{\partial p}f_1^\varepsilon-2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon\cdot\nabla_y f_2^\varepsilon$$

and the second order terms in r_j^ε, that is

$$\begin{aligned}&-\Delta_y\left[\frac{\partial v}{\partial p}(y,p)f_1^\varepsilon(t,y,p)-\left(\frac{\partial v}{\partial p}f_1\right)*\phi_\varepsilon(t,y,p)\right]f_2^\varepsilon\\ &-\Delta_y\left[\frac{\partial v}{\partial p}(y,p)f_2^\varepsilon(t,y,p)-\left(\frac{\partial v}{\partial p}f_2\right)*\phi_\varepsilon(t,y,p)\right]f_1^\varepsilon.\end{aligned}$$

In the following, we set

$$\varphi_i(t,y)=w(y,u_i(t,y))\quad(\text{i.e. } v(y,\varphi_i(t,y))=u_i(t,y)\ ),\tag{4.2}$$

$$\gamma_i(t,y)=\frac{1}{\frac{\partial v}{\partial p}(y,\varphi_i(t,y))}\left[\nabla_y u_i(t,y)-(\nabla_y v)(y,\varphi_i(t,y))\right]=\nabla_y\varphi_i(t,y).\tag{4.3}$$

We recall that

$$m_i(t,y,p) = |\nabla_y \varphi_i(t,y)|^2 \frac{\partial v}{\partial p}(y,\varphi_i(t,y))\ \delta(p=\varphi_i(t,y)) \tag{4.4}$$

$$= |\gamma_i|^2(t,y) \frac{\partial v}{\partial p}(y,\varphi_i(t,y))\ \delta(p=\varphi_i(t,y)), \tag{4.5}$$

$$\nabla_y f_i(t,y,p) = \eta_i \nabla_y \varphi_i(t,y)\delta(p=\varphi_i(t,y)) \tag{4.6}$$

$$= \eta_i \gamma_i(t,y)\delta(p=\varphi_i(t,y)) \tag{4.7}$$

$$\partial_p f_i = -\eta_i \delta(p=\varphi_i(t,y)), \tag{4.8}$$

for $i=1,2$, where $\eta_1 = 1$ and $\eta_2 = -1$.

First, for any test function $\theta = \theta(t,y) \in \mathcal{D}_{\mathrm{per}}([0,+\infty)\times Y)$ such that $\theta \geq 0$, for $\varepsilon < 1$, $R > 1$, we claim that

$$\begin{aligned}
&\int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p} f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon \cdot \nabla_y f_2^\varepsilon\right] \theta(t,y) K_R(p)\, dt\, dy\, dp \\
&\leq \int_0^\infty \int_{\mathbb{R}^{2N+2}} \int_{Y\times\mathbb{R}} \phi^\varepsilon(t-s_1, y-y_1, p-\varphi_1)\phi^\varepsilon(t-s_2, y-y_2, p-\varphi_2)\theta(t,y)K_R(p) \times \\
&\times 2\left[\gamma_1\cdot\gamma_2 \left(\frac{\partial v}{\partial p}(y,p) - \sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1)\frac{\partial v}{\partial p}(y_2,\varphi_2)}\right)\right]\, dy\, dp\, dy_1\, dy_2\, ds_1\, ds_2\, dt \\
&+2||\theta||_\infty \left[\mu_1(R-1) + \mu_1(-R+1) + \mu_2(R-1) + \mu_2(-R+1)\right]
\end{aligned} \tag{4.9}$$

In the integral of the right-hand side above, γ_i, φ_i are evaluated at (s_i, y_i) $(i=1,2)$.

The derivation of this inequality is rather technical, but straightforward if one follows the formal calculations of section 2. Let us focus on the first term of the left-hand side, namely

$$\begin{aligned}
I_\varepsilon &:= \int_0^\infty \int_{Y\times\mathbb{R}} \frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon \theta(t,y) K_R(p)\, dt\, dy\, dp \\
&= -\int_0^\infty\int_{Y\times\mathbb{R}} m_1^\varepsilon\ \partial_p f_2^\varepsilon\ \theta(t,y)K_R(p)\, dt\, dy\, dp - \int_0^\infty\int_{Y\times\mathbb{R}} m_1^\varepsilon f_2^\varepsilon\ \theta(t,y)\ K_R'(p)\, dt\, dy\, dp \\
&=: -(I_{\varepsilon,1} + I_{\varepsilon,2}).
\end{aligned}$$

Remembering (4.5) and (4.8), we have

$$m_1^\varepsilon(t,y,p) = \int_{\mathbb{R}^{N+1}} |\gamma_1(s_1,y_1)|^2 \frac{\partial v}{\partial p}(y_1,\varphi_1(s_1,y_1))\phi^\varepsilon(t-s_1, y-y_1, p-\varphi_1(s_1,y_1))\, ds_1\, dy_1,$$

$$\partial_p f_2^\varepsilon = \int_{\mathbb{R}^{N+1}} \phi^\varepsilon(t-s_2, y-y_2, p-\varphi_2(s_2,y_2))\, ds_2\, dy_2,$$

and thus

$$\begin{aligned}
I_{\varepsilon,1} = &\int_0^\infty \int_{\mathbb{R}^{2N+2}} \int_{Y\times\mathbb{R}} \phi^\varepsilon(t-s_1, y-y_1, p-\varphi_1)\phi^\varepsilon(t-s_2, y-y_2, p-\varphi_2)\theta(t,y)K_R(p) \times \\
&\times \left[\sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1(s_1,y_1))}\gamma_1(s_1,y_1)\right]^2\, dy\, dp\, dy_1\, dy_2\, ds_1\, ds_2\, dt.
\end{aligned}$$

On the other hand, according to lemma 4.1 and the assumptions on K_R,

$$\begin{aligned}
|I_{\varepsilon,2}| &\leq \int_0^\infty \int_{Y\times\mathbb{R}} m_1^\varepsilon \theta(t,y) |K_R'(p)|\, dt\, dy\, dp \\
&\leq 2||\theta||_\infty \left[\mu_1(R-1) + \mu_1(-R+1)\right].
\end{aligned}$$

The two other terms are treated in a similar way; eventually, we use the inequality $-(|a|^2+|b|^2) \le -2a\cdot b$ for all $a,b \in \mathbb{R}^N$ with $a = \sqrt{v_p(y_1,\varphi_1)}\,\gamma_1$, $b = \sqrt{v_p(y_2,\varphi_2)}\,\gamma_2$, and γ_i, φ_i are evaluated at $(s_i,y_i) \in [0,\infty)\times\mathbb{R}^N$. This concludes the derivation of (4.9).

Next, we compute, for any $\varphi_1,\varphi_2,p \in \mathbb{R}$, $y,y_1,y_2 \in Y$,

$$
\begin{aligned}
&\frac{\partial v}{\partial p}(y,p) - \sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1)\frac{\partial v}{\partial p}(y_2,\varphi_2)}\\
&= \frac{\sqrt{\frac{\partial v}{\partial p}(y_2,\varphi_2)}}{\sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1)}+\sqrt{\frac{\partial v}{\partial p}(y,p)}}\left[\frac{\partial v}{\partial p}(y,p) - \frac{\partial v}{\partial p}(y_1,\varphi_1)\right]\\
&\quad + \frac{\sqrt{\frac{\partial v}{\partial p}(y,p)}}{\sqrt{\frac{\partial v}{\partial p}(y_2,\varphi_2)}+\sqrt{\frac{\partial v}{\partial p}(y,p)}}\left[\frac{\partial v}{\partial p}(y,p) - \frac{\partial v}{\partial p}(y_2,\varphi_2)\right]\\
&= (y-y_1)\cdot B_1(y_1,y_2,y,\varphi_1,\varphi_2,p) + (y-y_2)\cdot B_2(y_1,y_2,y,\varphi_1,\varphi_2,p)\\
&\quad +(p-\varphi_1)b_1(y_1,y_2,y,\varphi_1,\varphi_2,p) + (p-\varphi_2)b_2(y_1,y_2,y,\varphi_1,\varphi_2,p)
\end{aligned}
$$

for some functions $B_1,B_2 \in \mathbb{R}^N$, $b_1,b_2 \in \mathbb{R}$ which can be computed in terms of $\frac{\partial v}{\partial p}$, $\nabla_y \frac{\partial v}{\partial p}$, and $\frac{\partial^2 v}{\partial p^2}$. Notice that hypothesis (2.8) ensures that $\frac{\partial^2 v}{\partial p^2}$ exists and is Hölder continuous in y, with locally uniform bounds in p (see theorem 8.24 in [9]), and hypothesis (2.9) entails that $\nabla_y \frac{\partial v}{\partial p}$ is Hölder continuous in y, with locally uniform bounds in p (see theorem 8.32 in [9]).
We denote by B_i^k the k-th component of B_i ($i=1,2$; $1\le k\le N$).

Now, set, for $(t,x,p) \in \mathbb{R}^{N+2}$,

$$
\begin{aligned}
\phi(t,x,p) &:= \zeta_1(t)\,\zeta_2(x)\,\zeta_3(p),\\
\phi_k(t,x,p) &:= x_k\,\phi(t,x,p), \quad 1\le k\le N,\\
\psi(t,x,p) &:= p\,\phi(t,x,p).
\end{aligned}
$$

Performing changes of variables in the right-hand side of (4.9), we are led to

$$
\begin{aligned}
&\int_0^\infty\int_{Y\times\mathbb{R}}\left[\frac{\partial m_1^\varepsilon}{\partial p}f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p}f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon\cdot\nabla_y f_2^\varepsilon\right]\theta(t,y)K_R(p)\,dt\,dy\,dp\\
\le\ & 2\int_0^\infty\int_{\mathbb{R}^{2N+2}}\int_{Y\times\mathbb{R}}\phi_k(s_1,y_1,p)\phi\left(s_2,y_2,p+\frac{\varphi_1-\varphi_2}{\varepsilon}\right)\theta(t,y)K_R(\varphi_1+\varepsilon p)\times\\
&\times\gamma_1\cdot\gamma_2\ B_1^k(y-\varepsilon y_1,y-\varepsilon y_2,y,\varphi_1,\varphi_2,\varphi_1+\varepsilon p)\\
&+2\int_0^\infty\int_{\mathbb{R}^{2N+2}}\int_{Y\times\mathbb{R}}\psi(s_1,y_1,p)\phi\left(s_2,y_2,p+\frac{\varphi_1-\varphi_2}{\varepsilon}\right)\theta(t,y)K_R(\varphi_1+\varepsilon p)\times\\
&\times\gamma_1\cdot\gamma_2\ b_1(y-\varepsilon y_1,y-\varepsilon y_2,y,\varphi_1,\varphi_2,\varphi_1+\varepsilon p)\\
&+2\int_0^\infty\int_{\mathbb{R}^{2N+2}}\int_{Y\times\mathbb{R}}\phi_k(s_2,y_2,p)\phi\left(s_1,y_1,p-\frac{\varphi_1-\varphi_2}{\varepsilon}\right)\theta(t,y)K_R(\varphi_2+\varepsilon p)\times\\
&\times\gamma_1\cdot\gamma_2\ B_2^k(y-\varepsilon y_1,y-\varepsilon y_2,y,\varphi_1,\varphi_2,\varphi_2+\varepsilon p)\\
&+2\int_0^\infty\int_{\mathbb{R}^{2N+2}}\int_{Y\times\mathbb{R}}\psi(s_2,y_2,p)\phi\left(s_1,y_1,p-\frac{\varphi_1-\varphi_2}{\varepsilon}\right)\theta(t,y)K_R(\varphi_2+\varepsilon p)\times\\
&\times\gamma_1\cdot\gamma_2\ b_2(y-\varepsilon y_1,y-\varepsilon y_2,y,\varphi_1,\varphi_2,\varphi_1+\varepsilon p)
\end{aligned}
$$

In each of the above integrals, γ_i and φ_i are now evaluated at $(t-\varepsilon s_i, y-\varepsilon y_i)$ (i=1,2).

Each of the integrals in the right-hand side of the above inequality is zero as soon as

$$|\varphi_1(t-\varepsilon s_1, y-\varepsilon y_1) - \varphi_2(t-\varepsilon s_2, y-\varepsilon y_2)| \geq 2\varepsilon;$$

and as $\varepsilon \to 0$, on the set $\{(t,y);\ \varphi_1(t,y) = \varphi_2(t,y)\}$,

$$B_1^k(y-\varepsilon y_1, y-\varepsilon y_2, y, \varphi_1, \varphi_2, \varphi_1+\varepsilon p) \to \frac{1}{2}\frac{\partial^2 v}{\partial p \partial y_k}(y, \varphi_1(t,y)),$$
$$b_1(y-\varepsilon y_1, y-\varepsilon y_2, y, \varphi_1, \varphi_2, \varphi_1+\varepsilon p) \to \frac{1}{2}\frac{\partial^2 v}{\partial p^2}(y, \varphi_1(t,y))$$

almost everywhere in $(t, s_1, s_2, y, y_1, y_2, p)$, and the same result holds for B_2^k and b_2 (as before, φ_1 and φ_2 are evaluated at $(t-\varepsilon s_i, y-\varepsilon y_i)$).

Hence, passing to the limit as $\varepsilon \to 0$, we obtain for all $R > 1$

$$\limsup_{\varepsilon\to 0} \int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p} f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon \cdot \nabla_y f_2^\varepsilon\right] \theta(t,y) K_R(\varphi_1+\varepsilon p)\, dt\, dy\, dp$$
$$\leq 2\int_0^\infty \int_Y \theta(t,y)\mathbf{1}_{\varphi_1=\varphi_2}\left[\lambda_k \frac{\partial^2 v}{\partial p\partial y_k}(y,\varphi_1) + \mu\frac{\partial^2 v}{\partial p^2}(y,\varphi_1)\right](\gamma_1\cdot\gamma_2)(t,y)K_R(\varphi_1(t,y))\, dt\, dy,$$

where

$$\lambda_k := \int_{\mathbb{R}^N} x_k\, \zeta_2(x)\, dx \int_{\mathbb{R}} \zeta_3(p)^2\, dp,$$
$$\mu := \int_{\mathbb{R}} p\, \zeta_3(p)^2\, dp.$$

At this stage, we could simply choose ζ_2 and ζ_3 such that $\lambda_k = 0\ \forall k$ and $\mu = 0$, but this would give the wrong impression that the limit depends on the choice of the regularization. In fact, we can investigate the limit coming from the second order terms in r_j^ε, i.e.

$$\int_0^\infty \int_{\mathbb{R}^{N+1}} -\left\{\Delta_y\left[\frac{\partial v}{\partial p}(y,p)f_1^\varepsilon(t,y,p) - \left(\frac{\partial v}{\partial p}f_1\right) * \phi_\varepsilon(t,y,p)\right] f_2^\varepsilon + \right.$$
$$\left. +\Delta_y\left[\frac{\partial v}{\partial p}(y,p)f_2^\varepsilon(t,y,p) - \left(\frac{\partial v}{\partial p}f_2\right) * \phi_\varepsilon(t,y,p)\right] f_1^\varepsilon\right\}\theta(t,y)K_R(p)\, dt\, dy\, dp.$$

With calculations similar to the ones lead above, using the identity (4.7), we obtain for instance

$$\lim_{\varepsilon\to 0}\int_0^\infty \int_{Y\times\mathbb{R}} -\Delta_y\left[\frac{\partial v}{\partial p}f_1^\varepsilon - \left(\frac{\partial v}{\partial p}f_1\right) * \phi_\varepsilon\right] f_2^\varepsilon\ \theta(t,y)K_R(p)\, dt\, dy\, dp$$
$$= \lim_{\varepsilon\to 0}\int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon - \left(\frac{\partial v}{\partial p}(\nabla_y f_1)\right) * \phi_\varepsilon\right]\cdot\nabla_y f_2^\varepsilon\ \theta(t,y)K_R(p)\, dt\, dy\, dp$$
$$= -\int_0^\infty \int_Y \lambda_k \frac{\partial^2 v}{\partial p\partial y_k}(y,\varphi_1)\mathbf{1}_{\varphi_1=\varphi_2}\gamma_1\cdot\gamma_2\ \theta(t,y)\ K_R(\varphi_1(t,y))\, dt\, dy$$
$$-\int_0^\infty \int_Y \mu\frac{\partial^2 v}{\partial p^2}(y,\varphi_1)\mathbf{1}_{\varphi_1=\varphi_2}\gamma_1\cdot\gamma_2\ \theta(t,y)\ K_R(\varphi_1(t,y))\, dt\, dy.$$

When deriving the second line from the first, we have used the fact that the other terms coming from the integration by parts all vanish as $\varepsilon \to 0$.

In a similar way, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^\infty \int_{Y\times\mathbb{R}} -\Delta_y \left[\frac{\partial v}{\partial p} f_2^\varepsilon - \left(\frac{\partial v}{\partial p} f_2\right) * \phi_\varepsilon\right] f_1^\varepsilon\ \theta(t,y) K_R(p)\ dt\ dy\ dp \\ = &-\int_0^\infty \int_Y \lambda_k \frac{\partial^2 v}{\partial p \partial y_k}(y,\varphi_2)\mathbf{1}_{\varphi_1=\varphi_2}\gamma_1\cdot\gamma_2\ \theta(t,y)\ K_R(\varphi_2(t,y))\ dt\ dy \\ &-\int_0^\infty \int_Y \mu\frac{\partial^2 v}{\partial p^2}(y,\varphi_2)\mathbf{1}_{\varphi_1=\varphi_2}\gamma_1\cdot\gamma_2\ \theta(t,y)\ K_R(\varphi_2(t,y))\ dt\ dy,\end{aligned}$$

so that eventually, for all test functions $\theta(t,y)\in\mathcal{D}_{\text{per}}([0,\infty)\times\mathbb{R}^N)$ such that $\theta\geq 0$,

$$\begin{aligned}\limsup_{R\to\infty}\limsup_{\varepsilon\to 0}\int_0^\infty\int_{Y\times\mathbb{R}}\left[\frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p} f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon\cdot\nabla_y f_2^\varepsilon + r_1^\varepsilon f_2^\varepsilon + r_2^\varepsilon f_1^\varepsilon\right]\times \\ \times\,\theta(t,y)K_R(p)dt\ dy\ dp \leq 0 \quad (4.10)\end{aligned}$$

Consequently, in the limit, we obtain for any test function $\theta(t,y)\in\mathcal{D}_{\text{per}}([0,\infty)\times\mathbb{R}^N)$ such that $\theta\geq 0$

$$\begin{aligned}&\int_0^\infty\int_Y (u_1-u_2)_+\partial_t\theta(t,y) + \mathbf{1}_{u_1>u_2}\left[A(y,u_1-A(y,u_2)\right]\cdot\nabla_y\theta(t,y)\ dtdy \\ \geq &\int_Y (u_1(t=0,y)-u_2(t=0,y))_+\ \theta(t=0,y)\ dy,\end{aligned}$$

which means exactly that

$$\partial_t(u_1-u_2)_+ + \text{div}_y\left[1_{u_1>u_2}\left(A(y,u_1)-A(y,u_2)\right)\right]\leq 0 \quad (4.11)$$

in the sense of distributions.

Integrating this last inequality on $(0,T)\times Y$ for any $T>0$ yields

$$||(u_1(t=T)-u_2(t=T))_+||_{L^1(Y)} \leq ||(u_1(t=0)-u_2(t=0))_+||_{L^1(Y)}. \quad (4.12)$$

Hence the derivation of (2.10) and (2.12) is complete; there only remains to prove lemma 4.1. The argument goes along the same lines as lemma 4.2.1 in [17].

Proof of Lemma 4.1. Notice that equation (4.1) is equivalent to

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_j\right) * \phi^\varepsilon + \frac{\partial}{\partial y_i}\left(\tilde{a}_i f_j\right) * \phi^\varepsilon - \Delta_y\left(\frac{\partial v}{\partial p} f_j\right) * \phi^\varepsilon = \eta_j\frac{\partial m_j^\varepsilon}{\partial p}. \quad (4.13)$$

Thus we focus on the derivation of (4.13) for f_1; let $(t,y,p)\in[0,\infty)\times Y\times\mathbb{R}$ be arbitrary. Following [17], one is tempted to consider the test function

$$(s,z,q)\mapsto\phi^\varepsilon(t-s,y-z,p-q)=\frac{1}{\varepsilon^{N+2}}\zeta_1\left(\frac{t-s}{\varepsilon}\right)\zeta_2\left(\frac{y-z}{\varepsilon}\right)\zeta_3\left(\frac{p-q}{\varepsilon}\right)$$

in the definition 2.3 of kinetic solutions. However, such a test function is not periodic in z, as required in definition 2.3; but the support of $z\mapsto\zeta_2\left((y-z)/\varepsilon\right)$ is a subset of $\bar{B}(y,\varepsilon)$, the closed ball centered on y and of radius ε. Thus, for $0<\varepsilon<1/2$,

$$\text{supp}\,\zeta_2\left((y-\cdot)/\varepsilon\right)\subset\bar{B}(y,\varepsilon)\subset\Pi_{i=1}^N\left(y_i-\frac{1}{2},y_i+\frac{1}{2}\right).$$

Hence for $\varepsilon < 1/2$, we can extend $\zeta_2\left((y-\cdot)/\varepsilon\right)$ by periodicity on $\mathbb{R}^N$; the function thus obtained is denoted by $\tilde{\zeta}_{y,\varepsilon}$, and belongs to $\mathcal{C}^\infty_{\text{per}}(\mathbb{R}^N)$.

Now, for fixed $(t, y, p) \in [0,\infty) \times Y \times \mathbb{R}$, we define the test function

$$\psi : (s, z, q) \mapsto \frac{1}{\varepsilon^{N+2}}\zeta_1\left(\frac{t-s}{\varepsilon}\right)\tilde{\zeta}_{y,\varepsilon}(z)\zeta_3\left(\frac{p-q}{\varepsilon}\right).$$

By construction, ψ belongs to $\mathcal{D}_{\text{per}}([0,\infty) \times Y \times \mathbb{R})$. Thus ψ is an admissible test function, and according to definition 2.3,

$$\int_0^\infty\int_{Y\times\mathbb{R}} f_1(s,z,q)\frac{\partial v(z,q)}{\partial q}\left\{\partial_s\psi + a_i(z, v(z,q))\partial_{z_i}\psi + \Delta_z\psi\right\} ds\, dz\, dq =$$
$$= \int_0^\infty\int_{Y\times\mathbb{R}} m(s,z,q)\partial_q\psi(s,z,q)\; dsdzdq - \int_{Y\times\mathbb{R}} \mathbf{1}_{u_0(z)>v(z,q)}\psi(0,z,q)\frac{\partial v(z,q)}{\partial q}dzdq.$$

First, notice that since $\operatorname{supp}\zeta_1 \subset [-1, 0]$, we have $\psi(0, z, q) = 0$ for all z, q. Moreover, since f_1 and ψ are Y-periodic in their second variable, we have for instance, setting $Y_y = \Pi_{i=1}^N\left(y_i - 1/2, y_i + 1/2\right) = y - e + Y$, where $e := (1/2, \cdots, 1/2) \in \mathbb{R}^N$,

$$\int_0^\infty\int_{Y\times\mathbb{R}} f_1\frac{\partial v}{\partial q}\partial_s\psi = \int_0^\infty\int_{Y_y\times\mathbb{R}} f_1\frac{\partial v}{\partial q}\partial_s\psi.$$

And when $z \in Y_y$, $\psi(s, z, q) = \phi^\varepsilon(t - s, y - z, p - q)$ by definition. Thus, using once again the assumption on the support of ζ_2,

$$\begin{aligned}
&\int_0^\infty\int_{Y\times\mathbb{R}} f_1(s,z,q)\frac{\partial v}{\partial q}(z,q)\partial_s\psi(s,z,q)ds\, dz\, dq\\
&= \int_0^\infty\int_{Y_y\times\mathbb{R}} f_1(s,z,q)\frac{\partial v}{\partial q}(z,q)\partial_s\phi^\varepsilon(t-s, y-z, p-q)ds\, dz\, dq\\
&= \int_0^\infty\int_{\mathbb{R}^N\times\mathbb{R}} f_1(s,z,q)\frac{\partial v}{\partial q}(z,q)\partial_s\phi^\varepsilon(t-s, y-z, p-q)ds\, dz\, dq\\
&= -\int_0^\infty\int_{\mathbb{R}^N\times\mathbb{R}} f_1(s,z,q)\frac{\partial v}{\partial q}(z,q)\partial_t\phi^\varepsilon(t-s, y-z, p-q)ds\, dz\, dq\\
&= -\partial_t\left[(f_1 v_p) * \phi^\varepsilon\right](t, y, p).
\end{aligned}$$

The other terms are treated in a similar way ; we obtain

$$\int_0^\infty\int_{Y\times\mathbb{R}} f_1(s,z,q)\tilde{a}_i(z,q)\partial_{y_i}\psi(s,z,q)\; ds\, dz\, dq =$$
$$=\int_0^\infty\int_{\mathbb{R}^N\times\mathbb{R}} f_1(s,z,q)\tilde{a}_i(z,q)\partial_{z_i}\phi^\varepsilon(t-s, y-z, p-q)\; ds\, dz\, dq = -\partial_{y_i}\left[f_1\tilde{a}_i\right] * \phi^\varepsilon(t, y, p),$$
$$\int_0^\infty\int_{Y\times\mathbb{R}} f_1(s,z,q)v_q(z,q)\Delta_z\psi(s,z,q)ds\, dz\, dq = \Delta_y\left[(f_1 v_p) * \phi^\varepsilon\right](t, y, p),$$
$$\int_0^\infty\int_{Y\times\mathbb{R}} m(s,z,q)\partial_q\psi(s,z,q)\; ds\, dz\, dq = -\partial_p m_1^\varepsilon(t, y, p).$$

There remains to derive the bound on m_1^ε : by definition,

$$
\begin{aligned}
&\int_0^\infty \int_Y m_1^\varepsilon(t,y,p)\,dt\,dy\\
&= \int_0^\infty \int_Y \int_{\mathbb{R}^{N+2}} m_1(s,z,q)\phi^\varepsilon(t-s,y-z,p-q)ds\,dz\,dq\,dt\,dy\\
&= \int_Y \int_{\mathbb{R}^{N+1}} \int_0^\infty \int_0^\infty m_1(s,z,q)\phi^\varepsilon(t-s,y-z,p-q)\,dt\,ds\,dz\,dq\,dy\\
&= \int_Y \int_{\mathbb{R}^{N+1}} \int_0^\infty \int_{-s}^\infty m_1(s,z,q)\phi^\varepsilon(u,y-z,p-q)\,du\,ds\;\,dz\,dq\,dy\\
&\le \int_Y \int_{\mathbb{R}^{N+1}} \int_0^\infty \int_{\mathbb{R}} m_1(s,z,q)\phi^\varepsilon(u,y-z,p-q)\,du\,ds\;\,dz\,dq\,dy\\
&\le \frac{1}{\varepsilon^{N+1}} \int_Y \int_{\mathbb{R}^{N+1}} \int_0^\infty m_1(s,z,q)\zeta_2\left(\frac{y-z}{\varepsilon}\right)\zeta_3\left(\frac{p-q}{\varepsilon}\right)\,ds\;\,dz\,dq\,dy.
\end{aligned}
$$

Then, with the same notations as earlier,

$$
\begin{aligned}
\int_Y \int_{\mathbb{R}^N} m_1(s,z,q)\zeta_2\left(\frac{y-z}{\varepsilon}\right)\,dz\,dy &= \int_Y dy \left(\int_{Y_y} dz\; m_1(s,z,q)\zeta_2\left(\frac{y-z}{\varepsilon}\right)\right)\\
&= \int_{Y\times Y} m_1(s,y+y'-e,q)\zeta_2\left(\frac{-y'+e}{\varepsilon}\right)dy\,dy'\\
&= \int_{Y\times Y} m_1(s,y,q)\zeta_2\left(\frac{-y'+e}{\varepsilon}\right)dy\,dy'\\
&= \left(\int_Y m_1(s,y,q)\,dy\right)\times\left(\int_{\mathbb{R}^N} \zeta_2\left(\frac{-y'}{\varepsilon}\right)\,dy'\right).
\end{aligned}
$$

In the one before last step, we have used the periodicity of m_1.

Thus

$$
\begin{aligned}
\int_0^\infty \int_Y m_1^\varepsilon(t,y,p)\,dt\,dy &\le \frac{1}{\varepsilon}\int_{\mathbb{R}} \int_{z\in Y} \int_0^\infty m_1(s,z,q)\zeta_3\left(\frac{p-q}{\varepsilon}\right)\,ds\;\,dz\,dq\\
&\le \frac{1}{\varepsilon}\int_{\mathbb{R}} \mu_1(q)\zeta_3\left(\frac{p-q}{\varepsilon}\right)\,dq\\
&\le \int_{-1}^1 \mu_1(p-\varepsilon q)\zeta_3(q)\,dq.
\end{aligned}
$$

The monotonicity of μ_1 yields the desired result.

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