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Initial layer for the homogenization of a conservation law with vanishing viscosity

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Abstract

We study the limit as $\varepsilon \to 0$ of the solutions of the equation $\partial_t u^\varepsilon + \mathrm{div}_x \left[A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\right] - \varepsilon \Delta_x u^\varepsilon = 0$. This problem was already addressed in a previous article in the case of well-prepared initial data, i.e. when the microscopic profile of the solution is adapted to the medium at time $t = 0$. Here, we prove that when the initial data is not well-prepared, there is an initial layer during which the solution adapts itself to match the profile dictated by the environment. The typical size of the initial layer is of order ε. The proof relies strongly on the parabolic form of the equation; in particular, no condition of nonlinearity on A is required.

Key words. Homogenization – Parabolic scalar conservation law– Initial layer

1. Introduction

We study the homogenization of equation

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) - \varepsilon \Delta u^\varepsilon = 0 \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{1}$$

$$u^\varepsilon(t=0) = u_0\left(x, \frac{x}{\varepsilon}\right). \tag{2}$$

It is well-known (see for instance [9,10]) that under suitable regularity assumptions on A and u_0, there exists a unique solution $u^\varepsilon(t,x)$ of (1) in $\mathcal{C}([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^\infty_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^N) \cap L^2_{\mathrm{loc}}([0,\infty), H^1_{\mathrm{loc}}(\mathbb{R}^N))$. The first part of the homogenization process was already performed in [2], the main results of which we recall below. Precisely, it is proved in [2] that if the initial data is already adapted to the microstructure, then $u^\varepsilon(t,x)$ behaves in $L^2_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^N)$ like some function $v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$, where $v(y,p)$ is the solution of a microscopic cell problem depending on the parameter $p \in \mathbb{R}$, and $\bar{u}(t,x)$ is the entropy solution of a nonlinear scalar conservation law. This result was proved with the help of two-scale Young measures, a tool introduced by Weinan E in [3]. Although equation (1) is parabolic, the proof bears a lot of resemblance with the ones of Weinan E in [3] and Weinan E and Denis Serre in [5], both of which tackle hyperbolic problems. However, this is not surprising if we take into account the scaling of the viscosity: indeed, since the viscosity is of order ε, it has an effect on the microscopic asymptotic profile of the solution u^ε, but it disappears from the macroscopic homogenized problem, which is hyperbolic.

In this article, we go one step further than Weinan E and Denis Serre in [3], [5] (see also [4]), since we are able to prove that homogenization holds even when the initial data is not well-prepared, i.e. when it cannot be written as $u_0\left(x, \frac{x}{\varepsilon}\right) = v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)$ for some function $\bar{u}_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$. In that case, there is an initial layer of order ε during which the solution adapts itself to the microstructure. The proof relies strongly on the parabolic form of the equation, which compels the solutions of (1) to match the microscopic profile $v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$ exponentially fast. In the case when the viscosity is zero, nonlinearity assumptions on the flux A are most probably necessary in order to obtain the same kind of result, but few cases are known: in [6], Bjorn Engquist and Weinan E prove that homogenization holds in dimensions one and two for a nonlinear homogeneous flux in the case when the initial data is oscillating. In [4], Weinan E studies a particular kind of heterogeneous conservation law in dimension one, for which he proves a result similar to our theorem 1 under a strict convexity assumption (see also [3] for further results in the linear case). Let us point out that this is certainly linked to the compactness of solutions of conservation laws and to the cancellation of oscillations in the case when the flux is nonlinear (see [15], [14]). However, even if this connection seems natural, it is still an open problem how to handle initial layers (or for that matter, homogenization in general) in the hyperbolic case when the dimension is greater than or equal to two (for N=1, an equivalence with Hamilton-Jacobi equations allows us to use the results of P.-L. Lions, G. Papanicolaou and S.R.S. Varadhan in [13], and thus to identify the weak limit of u^ε as $\varepsilon \to 0$).

Throughout this article, we use the notation $Y := \Pi_{i=1}^N (0, T_i)$, $T_i > 0$ for $1 \le i \le N$ (Y is the unit cell), and

$$\langle v \rangle := \frac{1}{|Y|} \int_Y v(y)\, dy;$$

we will work in the following functional spaces: if $\mathcal{C}^\infty_{\text{per}}(Y)$ denotes the space of Y-periodic functions in $\mathcal{C}^\infty(\mathbb{R}^N)$, then:

$$\begin{gathered} H^1_{\text{per}}(Y) := \overline{\mathcal{C}^\infty_{\text{per}}(Y)}^{H^1(Y)}, \ ||\cdot||_{H^1_{\text{per}}(Y)} = ||\cdot||_{H^1(Y)}, \\ V := \{v \in H^1_{\text{per}}(Y), \langle v \rangle_Y = 0\}, \ ||v||_V = ||\nabla v||_{L^2(Y)} \\ \mathcal{C}^\infty_{\text{per}}(Y \times \mathbb{R}) := \{f = f(y,v) \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}); f \text{ is } Y-\text{periodic in } y\}, \\ W^{k,\infty}_{\text{per}}(Y \times \mathbb{R}) := \overline{\mathcal{C}^\infty_{\text{per}}(Y \times \mathbb{R})}^{W^{k,\infty}(Y\times\mathbb{R})}, \quad k \in \mathbb{N}, \\ W^{1,\infty}_{\text{per,loc}}(Y \times \mathbb{R}) := \{u = u(y,v) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^{N+1}), u \text{ is } Y-\text{periodic in } y\}. \end{gathered}$$

Thanks to the Poincaré-Wirtinger inequality, the norm on V is equivalent to the H^1 norm.

We will often use the following notations:

$$a_i(y,v) := \frac{\partial A_i(y,v)}{\partial v} \quad (1 \le i \le N), \quad a_{N+1}(y,v) := -\sum_{i=1}^N \frac{\partial A_i(y,v)}{\partial y_i}.$$

Let us now recall the main results of [2]; the first one is about the cell problem:

Proposition 1. *Let $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$. Assume that there exist $C_0 > 0$, $m \in [0,\infty)$, $n \in [0, \frac{N+2}{N-2})$ when $N > 2$, such that for all $(y,p) \in Y \times \mathbb{R}$*

$$|a_i(y,p)| \le C_0 \left(1 + |p|^m\right) \quad \forall\, 1 \le i \le N, \tag{3}$$
$$|a_{N+1}(y,p)| \le C_0 \left(1 + |p|^n\right). \tag{4}$$

Assume as well that at least one of the following conditions holds:

$$m = 0 \tag{5}$$
$$or \quad 0 \le n < 1 \tag{6}$$
$$or \quad n < \frac{N+2}{N} \ and\ \exists p_0 \in \mathbb{R}\ s.t.\ a_{N+1}(y,p_0) = 0\ \forall y \in Y. \tag{7}$$

Then for all $p \in \mathbb{R}$, there exists a unique solution $\tilde{u} \in V$ of the cell problem

$$-\Delta_y \tilde{u} + \operatorname{div}_y A(y, p + \tilde{u}) = 0; \tag{8}$$

For all $p \in \mathbb{R}$, $\tilde{u}(\cdot,p)$ belongs to $W^{2,q}_{per}(Y)$ for all $1 < q < +\infty$ and satisfies the following a priori estimate for all $R > 0$

$$||\tilde{u}(\cdot,p)||_{W^{2,q}(Y)} \le C \quad \forall p \in \mathbb{R}, |p| \le R, \tag{9}$$

for some constant C depending only on N, Y, C_0, m, n, q and R.

Moreover, setting $v(y,p) := p + \tilde{u}(y,p)$, the sequence v is increasing in p: for every $p > p'$,

$$v(y,p) > v(y,p') \quad \forall y \in Y.$$

The homogenization result proved in [2] is stated in the following

Proposition 2. *Assume that $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$ satisfies the hypotheses of proposition 1, and that $\frac{\partial a_i}{\partial y_j} \in L^\infty_{loc}(Y \times \mathbb{R})$, $\frac{\partial a_i}{\partial v} \in L^\infty_{loc}(Y \times \mathbb{R})$ for $1 \le i \le N+1$, $1 \le j \le N$.*

Let $p \in \mathbb{R}$, and let $\tilde{u}$ be the unique solution in V of the cell problem (8).

Let

$$\bar{A}_i(p) := \frac{1}{|Y|} \int_Y A(y, p + \tilde{u}(y,p))\, dy. \tag{10}$$

Assume also that u_0 is "well-prepared", i.e. satisfies

$$u_0(x,y) = v(y, \bar{u}_0(x)) \tag{11}$$

for some $\bar{u}_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$.

Then as ε goes to 0,

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0 \quad \text{in } L^2_{loc}([0,\infty) \times \mathbb{R}^N),$$

where $\bar{u} = \bar{u}(t,x) \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty) \times \mathbb{R}^N)$ is the unique entropy solution of the hyperbolic scalar conservation law

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \displaystyle\sum_{i=1}^N \dfrac{\partial \bar{A}_i(\bar{u}(t,x))}{\partial x_i} = 0, \\ \bar{u}(t=0,x) = \bar{u}_0(x) \in L^1 \cap L^\infty(\mathbb{R}^N). \end{cases} \tag{12}$$

We now state the main results of this paper; the first one addresses the long-time behavior of the solutions of a parabolic problem, which is derived by inserting in (1) a two-scale Ansatz in both space and time, namely

$$u^\varepsilon(t,x) \approx u^0\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \cdots$$

Theorem 1. *Let $u_0 \in L^\infty(Y)$. Assume that $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$ satisfies the hypotheses of proposition 1, and that $\frac{\partial a_i}{\partial y_j} \in L^\infty_{loc}(Y \times \mathbb{R})$. Assume also that there exist constants $A, B \in \mathbb{R}$ such that*

$$v(y,A) \le u_0(y) \le v(y,B). \tag{13}$$

Let $u \in \mathcal{C}([0,\infty), L^1(Y)) \cap L^\infty([0,\infty) \times Y) \cap L^2_{loc}([0,\infty), H^1_{per}(Y))$ be the unique solution of the parabolic equation

$$\begin{cases} \dfrac{\partial u(\tau,y)}{\partial \tau} + \operatorname{div}_y [A(y, u(\tau,y))] - \Delta_y u(\tau,y) = 0, \quad \tau \ge 0,\ y \in Y, \\ u(\tau=0,y) = u_0(y). \end{cases} \tag{14}$$

Let $p = \langle u_0 \rangle$, and let $v(y,p) \in H^1_{per}(Y)$ be the solution of the associated cell problem (8).

Then as $\tau \to \infty$,

$$||u(\tau,y) - v(y,p)||_{L^\infty(Y)} \to 0. \tag{15}$$

Consequently, there exist constants $c, \mu > 0$, with μ depending only on Y, n and $\max_{1 \le i \le N} ||a_i||_{L^\infty(Y \times (-K,K))}$ where $K = \sup_{A \le p \le B} ||v(\cdot,p)||_{L^\infty(Y)}$ such that

$$||u(\tau,y) - v(y,p)||_{L^\infty(Y)} \le c||u_0(y) - v(y,p)||_{L^2(Y)} e^{-\mu\tau} \quad \forall \tau \ge 1. \tag{16}$$

The proof of the first part of theorem 1, i.e. of the convergence result (15), is given is section 2 and relies strongly on the parabolic form of equation (14). Thus, no condition of nonlinearity on A is required. The same kind of result has been proved for hyperbolic scalar conservation laws under strict nonlinearity conditions (see [6], [12], [15] and the references therein). The second part of the theorem, i.e. the exponential decay result stated in (16), will be a straightforward consequence of one of the lemmas in section 3.

Combining the results of [2] and of theorem 1, we obtain the following homogenization result:

Theorem 2. *Let $u_0 \in L^1_{loc}(\mathbb{R}^N; \mathcal{C}_{per}(Y))$ such that there exist constants $A, B \in \mathbb{R}$ such that*

$$v(y, A) \leq u_0(x, y) \leq v(y, B) \quad \text{for a.e. } x \in \mathbb{R}^N, y \in Y. \tag{17}$$

Assume that A satisfies the hypotheses of theorem 1 and that $\partial_v a_i(y, \cdot) \in \mathcal{C}(\mathbb{R})$ for a.e. $y \in Y$ and for $1 \leq i \leq N+1$. Then for all $0 < a < b$, for all $R > 0$

$$\left|\left| u^\varepsilon(t, x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t, x)\right) \right|\right|_{L^1((a,b)\times B_R)} \to 0 \quad \text{as } \varepsilon \to 0,$$

where $\bar{u}$ is the solution of the homogenized problem (12) with initial data

$$\bar{u}(t = 0, x) = \bar{u}_0(x) = \langle u_0(x, \cdot) \rangle .$$

Remark 1. Notice that hypothesis (13) (or hypothesis (17)) is somehow a generalization of the well-preparedness hypothesis on the initial data in proposition 2. It implies in particular that $u^0 \in L^\infty(\mathbb{R}^N \times Y)$. Conversely, if u^0 is any function in $L^\infty(\mathbb{R}^N \times Y)$, and if

$$\begin{aligned} &\lim_{p\to+\infty} \inf_Y v(y, p) = +\infty, \\ &\lim_{p\to-\infty} \sup_Y v(y, p) = -\infty, \end{aligned} \tag{18}$$

then we can always find constants $A, B \in \mathbb{R}$ such that (17) is satisfied. And (18) is always satisfied when $a_i \in L^\infty(Y \times \mathbb{R})$ for $1 \leq i \leq N$ (see [2]).

However, when (18) is not satisfied (see [2] for examples in which this condition is violated), we could not find more general hypothesis on the initial data. Basically, hypothesis (17) provides a subsolution and a supersolution of (1) which are bounded uniformly in ε in $L^\infty((0, T) \times \mathbb{R}^N)$. In particular, this implies uniform L^∞ bounds on u^ε, which are not easy to derive otherwise.

2. Exponential convergence towards solutions of the cell problem - proof of theorem 1

This section is devoted to the proof of theorem 1; the proof uses Harnack's inequality and therefore relies strongly on the parabolic form of equation (14). Equation (14) is derived by means of a formal double-scale expansion in time and space variables :

$$u^\varepsilon(t, x) = u^0\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \cdots$$

Inserting this expansion in equation (1) and identifying the coefficient of ε^{-1} yields equation (14).

Before tackling the proof of theorem 1, let us recall a few facts about the solutions of equations (14) and (8):

1. $v(y, p) \geq v(y, p')$ for all $y \in Y$ for $p \geq p'$;
2. $v \in \mathcal{C}(\mathbb{R}_p, \mathcal{C}^{0,\gamma}(\overline{Y}))$ for some $\gamma \in (0, 1)$;
3. If u_1, u_2 are solutions of (14), then for $0 \leq s \leq t$

$$||u_1(t) - u_2(t)||_{L^1(Y)} \leq ||u_1(s) - u_2(s)||_{L^1(Y)}, \tag{19}$$

$$||\left(u_1(t) - u_2(t)\right)_+||_{L^1(Y)} \leq ||\left(u_1(s) - u_2(s)\right)_+||_{L^1(Y)}; \tag{20}$$

4. If u is a solution of (14) with initial data satisfying (13), then $u \in L^\infty([0,\infty)\times Y)$ and $v(y,A) \leq u(\tau,y) \leq v(y,B)$ for all $\tau \geq 0,\ y \in Y$.

We skip the proof of these properties; the first two are proved in [2]. In particular, the second one follows from the fact that $v \in L^\infty_{\text{loc}}(\mathbb{R}_p; W^{2,q}(Y)) \cap \mathcal{C}(\mathbb{R}_p; H^1_{\text{per}}(Y))$ for all $q \in [1,\infty)$. (19) and (20) can be shown by Kruzkhov's method (see [17] for instance in the case of a hyperbolic homogeneous conservation law). The last property is a consequence of (20).

We are now ready to prove theorem 1 : define, for $y \in Y$, $t \in [0,\infty)$

$$U(t,y) := \sup_{\tau \geq t} u(\tau,y),$$
$$\text{and } p^*(t) := \inf\{p; v(y,p) \geq U(t,y) \text{ for a.e. } y \in Y\} \leq B < \infty.$$

Then it is easily proved that

1. $U \in L^\infty([0,\infty) \times Y)$;
2. $U(t,y) \leq U(t',y)$ for $t \geq t'$ and for a.e. $y \in Y$;
3. $v(y,p^*(t)) \geq U(t,y)$ for all $t \in [0,\infty)$, for a.e. $y \in Y$;
4. $p^*(t)$ is a bounded non-increasing function of t; let $p^* := \lim_{t\to\infty} \downarrow p^*(t)$;
5. for all $t \in [0,\infty)$, if $p < p^*(t)$, then the set
$$E(p,t) := \{y \in Y; v(y,p) < U(t,y)\}$$
has positive measure.

Set $\delta > 0$; since v is a continous function of p, there exists $\eta > 0$ such that

$$|p - p^*| \leq \eta \Rightarrow ||v(\cdot,p) - v(\cdot,p^*)||_{L^\infty(Y)} \leq \delta;$$

take $t_0 \in \mathbb{R}$ such that if $t \geq t_0$, then $|p^* - p^*(t)| \leq \eta$, and take $p_\delta < p^*$ such that $|p_\delta - p^*| \leq \eta$. Then

$$||v(\cdot,p^*(t)) - v(\cdot,p_\delta)||_{L^\infty(Y)} \leq 2\delta \quad \forall t \geq t_0$$

and the set $E(p_\delta,t)$ has positive measure for all $t \geq 0$. Hence, for $t \geq t_0$, for $y \in E(p_\delta,t)$, we have

$$v(y,p^*(t)) - 2\delta \leq v(y,p_\delta) \leq U(t,y) \leq v(y,p^*(t)). \tag{21}$$

Now, take any sequence $t_n \to \infty$, and for all $n \in \mathbb{N}$ choose $y_n^\delta \in E(p_\delta, t_n+1)$; there exists $\tau_n^\delta \geq t_n + 1$ such that

$$\left|u(\tau_n^\delta, y_n^\delta) - U(t_n+1, y_n^\delta)\right| \leq \delta; \tag{22}$$

then for n large enough, $t_n \geq t_0$ and gathering (21) and (22) leads us to

$$v(y_n^\delta,p^*(t_n)) - 2\delta \leq v(y_n^\delta,p_\delta) \leq U(t_n+1,y_n^\delta) \leq v(y_n^\delta,p^*(t_n+1)) \leq v(y_n^\delta,p^*(t_n)),$$
$$\left|u(\tau_n^\delta,y_n^\delta) - v(y_n^\delta,p^*(t_n))\right| \leq 3\delta.$$

Set, for $s \in [-1,1]$ and $y \in Y$,

$$w_n^\delta(s,y) := v(y,p^*(t_n)) - u(\tau_n^\delta + s, y);$$

since $\tau_n^\delta \geq t_n + 1$, according to the definition of $p^*(t_n)$ and to that of $U(t_n)$, w_n^δ is a nonnegative function for all $n \in \mathbb{N}, \delta > 0$. Moreover, thanks to our preliminary analysis, for n large enough,

$$w_n^\delta(s=0,y_n^\delta) \leq 3\delta.$$

w_n^δ is therefore a nonnegative solution of the parabolic equation

$$\frac{\partial w_n^\delta}{\partial s} + \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(b_i^{n,\delta}(s,y) w_n^\delta(s)\right) - \Delta_y w_n^\delta = 0,$$

where

$$b_i^{n,\delta}(s,y) := \int_0^1 a_i(y, \tau v(y,p^*(t_n)) + (1-\tau) u(\tau_n^\delta + s, y))\, d\tau.$$

Let $K>0$ such that

$$-K \le v(y,A) \le v(y,B) \le K \quad \forall y \in Y.$$

Then

$$||b_i^{n,\delta}||_{L^\infty((-1,1)\times Y)} \le ||a_i||_{L^\infty(Y\times(-K,K))} \quad \forall n \in \mathbb{N}\ \forall \delta>0.$$

According to Harnack's inequality, there exists a constant C depending only on Y and $||a||_{L^\infty(Y\times(-K,K))^N}$ such that for n large enough

$$\sup_{y\in Y} w_n^\delta(-\frac{1}{2},y) \le C \inf_{y\in Y} w_n^\delta(0,y). \tag{23}$$

Hence, we have proved that for n large enough,

$$0 \le v(y,p^*(t_n)) - u(\tau_n^\delta - \frac{1}{2}, y) \le C\delta \quad \forall n \in \mathbb{N}\ \forall y \in Y,$$

But as $n \to \infty$

$$v(y,p^*(t_n)) \to v(y,p^*) \quad \text{in } L^\infty(Y).$$

Thus there exists a sequence $T_n \to \infty$ such that

$$||u(T_n,y) - v(y,p^*)||_{L^\infty(Y)} \to 0 \tag{24}$$

and the first part of the theorem is thereby proved. It remains to prove that $p^* = \langle u_0 \rangle$ and that

$$u(t,\cdot) \to v(\cdot,p^*)$$

in $L^q(Y)$ if $1 \le q < \infty$; since $\int_Y u(t)$ is conserved by the equation, as $n \to \infty$

$$\langle u(T_n)\rangle = \langle u_0\rangle \to \langle v(\cdot,p^*)\rangle = p^*,$$

and $p^* = \langle u_0\rangle$. Using the L^1 contraction property (19) for equation (14) with $u_1 = u$, $u_2 = v(y,p^*)$ (u_2 is a stationary solution of (14)), $s = T_n$ and $t \ge T_n$ shows that

$$u(t,\cdot) \to v(\cdot,p^*) \quad \text{in } L^1(Y).$$

Interpolating this convergence result with the fact that $u(t,y) - v(y,p^*)$ is bounded in $L^\infty([0,\infty)\times Y)$ yields the strong convergence in $L^q(Y)$ for all $q \in [1,\infty)$, and thus in $L^\infty(Y)$ by parabolic regularity. □

3. Homogenization of equation (1) in case of ill-prepared initial data

The proof of theorem 2 is rather technical, but the ideas involved are quite simple. Our guess is that u^ε behaves in L^1_{loc} as

$$z^\varepsilon := w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) + v^\varepsilon(t,x)$$

where

- $w(\tau,x,y)$ is the solution of (14) with initial data $u_0(x,y)$ (x being treated as a parameter),
- $\bar{u}$ is the solution of the homogenized problem with initial data $\bar{u}_0(x) := \langle u_0(x,\cdot)\rangle$,
- $v^\varepsilon(t,x)$ is the solution of (1) with initial data $v^\varepsilon(t=0,x) = v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)$.

Let us point out the roles of the different terms in z^ε : for small times, v^ε is close to $v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)$, and thus $z^\varepsilon \approx w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) \approx w\left(\tau = 0, x, \frac{x}{\varepsilon}\right)$. On the contrary, on time scales which are large with respect to ε, $w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) = \mathcal{O}(e^{-\frac{\mu t}{\varepsilon}})$, and thus $z^\varepsilon \approx v^\varepsilon \approx v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$. In other words, there is an initial layer of size ε which is described by $w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right)$, and after which z^ε behaves as $v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$.

In order to prove that $u^\varepsilon - z^\varepsilon$ goes to 0 in $L^1_{\text{loc}}((0,+\infty)\times\mathbb{R}^N)$, we could compute

$$f^\varepsilon := \frac{\partial z^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, z^\varepsilon(t,x)\right) - \varepsilon \Delta z^\varepsilon$$

and prove that f^ε goes to 0 in $L^1_{\text{loc}}((0,+\infty)\times\mathbb{R}^N)$ (notice that $u^\varepsilon(t=0) = z^\varepsilon(t=0)$). However, this involves rather heavy and unnecessary calculations. Instead, we notice that since u^ε and v^ε are both solutions of (1), we can use the L^1 contraction principle:

$$\partial_t |u^\varepsilon - v^\varepsilon| + \sum_{i=1}^N \frac{\partial}{\partial x_i} \eta_i\left(\frac{x}{\varepsilon}, u^\varepsilon, v^\varepsilon\right) - \varepsilon \Delta_x |u^\varepsilon - v^\varepsilon| \le 0, \tag{25}$$

where

$$\eta_i(y,v,w) := \text{sgn}(v-w)\left[A_i(y,v) - A_i(y,w)\right].$$

First, let us prove that u^ε and v^ε are uniformly bounded in L^∞: since equation (1) is order preserving, for all $t,\tau \ge 0$, for a.e. $x \in \mathbb{R}^N,\ y \in Y$ we have

$$v\left(\frac{x}{\varepsilon}, A\right) \le u^\varepsilon(t,x) \le v\left(\frac{x}{\varepsilon}, B\right) \quad \forall t \ge 0 \text{ for a.e. } x \in \mathbb{R}^N,$$
$$A \le \bar{u}_0(x) \le B \quad \text{for a.e. } x \in \mathbb{R}^N,$$
$$v\left(\frac{x}{\varepsilon}, A\right) \le v^\varepsilon(t,x) \le v\left(\frac{x}{\varepsilon}, B\right) \quad \forall t \ge 0 \text{ for a.e. } x \in \mathbb{R}^N.$$

Consequently, there exists a constant $K > 0$ depending only on A, B, N, Y, n, m such that

$$|u^\varepsilon(t,x)| + |v^\varepsilon(t,x)| \le K \quad \forall t \ge 0 \text{ for a.e. } x \in \mathbb{R}^N.$$

Now, let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $\varphi(x) = e^{-|x|}$ when $|x| \ge 1$, and $\frac{1}{e} \le \varphi(x) \le 1$ for $|x| \le 1$. Notice that it is enough to prove that

$$\int_a^b \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - v^\varepsilon(t,x)|\ \varphi(x)\, dx\, dt \to 0$$

as $\varepsilon \to 0$ for all $0 < a < b$. Moreover, there exists a constant C such that

$$|\nabla_x \varphi(x)|, |\Delta_x \varphi(x)| \le C\varphi(x) \quad \forall x \in \mathbb{R}^N.$$

Hence, multiplying (25) by $\varphi(x)$ and integrating on $(\alpha, T)\times\mathbb{R}^N$ yields

$$\begin{aligned}\int_{\mathbb{R}^N} |u^\varepsilon(T,x) - v^\varepsilon(T,x)|\varphi(x)\ dx \le& \int_{\mathbb{R}^N} |u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)|\varphi(x)\ dx \\ &+C\int_\alpha^T \int_{\mathbb{R}^N} \left|\eta\left(\frac{x}{\varepsilon}, u^\varepsilon, v^\varepsilon\right)\right| \varphi(x)\ dx\, dt \\ &+\varepsilon C \int_\alpha^T \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - v^\varepsilon(t,x)|\varphi(x)\ dx\, dt\end{aligned}$$

Thanks to the L^∞ bound, we deduce that

$$\left|\eta_i\left(\frac{x}{\varepsilon}, u^\varepsilon, v^\varepsilon\right)\right| \le ||a_i||_{L^\infty(Y\times(-K,K))} |u^\varepsilon - v^\varepsilon|$$

and thus for all $\varepsilon \in (0,1)$

$$\int_{\mathbb{R}^N} |u^\varepsilon(T,x) - v^\varepsilon(T,x)|\varphi(x)\ dx \leq \int_{\mathbb{R}^N} |u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)|\varphi(x)\ dx$$
$$+C\int_\alpha^T \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - v^\varepsilon(t,x)|\varphi(x)\ dx\, dt$$

where the constant C depends only on $||a_i||_{L^\infty(Y\times(-K,K))}$. Using Gronwall's lemma, we deduce that

$$\int_{\mathbb{R}^N} |u^\varepsilon(T,x) - v^\varepsilon(T,x)|\varphi(x)\ dx \leq e^{C(T-\alpha)} \int_{\mathbb{R}^N} |u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)|\varphi(x)\ dx.$$

On the other hand, for all $R > 1$, we have

$$\int_{\mathbb{R}^N} |u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)|\varphi(x)\ dx \leq ||u^\varepsilon(\alpha) - v^\varepsilon(\alpha)||_{L^1(B_R)} + Ce^{-R},$$

where C depends only on N and K.

Thus, for all $R > 1$

$$\int_{\mathbb{R}^N} |u^\varepsilon(T,x) - v^\varepsilon(T,x)|\varphi(x)\ dx \leq e^{C(T-\alpha)} \left[||u^\varepsilon(\alpha) - v^\varepsilon(\alpha)||_{L^1(B_R)} + Ce^{-R} \right]. \tag{26}$$

It remains to prove that for all $R > 0$, we can choose ε and α small enough so that $||u^\varepsilon(\alpha) - v^\varepsilon(\alpha)||_{L^1(B_R)}$ is arbitrarily small. But for small α, our guess is that $u^\varepsilon(\alpha,x)$ behaves as $w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right)$. If we follow this intuition, we are led to

$$\begin{aligned}
||u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)||_{L^1(B_R)} \leq& \left|\left| u^\varepsilon(\alpha,x) - w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right) \right|\right|_{L^1(B_R)} \\
&+ \left|\left| w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) \right|\right|_{L^1(B_R)} \\
&+ \left|\left| v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right) \right|\right|_{L^1(B_R)} \\
&+ \left|\left| v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right) - v^\varepsilon(\alpha,x) \right|\right|_{L^1(B_R)} \\
\leq& \left|\left| u^\varepsilon(\alpha,x) - w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right) \right|\right|_{L^1(B_R)} \\
&+ \left|\left| w\left(\frac{\alpha}{\varepsilon}, x, y\right) - v\left(y, \bar{u}_0(x)\right) \right|\right|_{L^1(B_R;\mathcal{C}_{\text{per}}(Y))} \\
&+ ||v\left(y, \bar{u}_0(x)\right) - v\left(y, \bar{u}(\alpha,x)\right)||_{L^1(B_R;\mathcal{C}_{\text{per}}(Y))} \\
&+ \left|\left| v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right) - v^\varepsilon(\alpha,x) \right|\right|_{L^1(B_R)}.
\end{aligned}$$

In the next subsections we prove that each of the terms of the right-hand side of the above inequality goes to 0 as $\alpha \to 0$ and $\varepsilon \to 0$.

We have used above the following proposition, due to Allaire ([1]):

Proposition 3. *Let Ω be an open set of $\mathbb{R}^N$.*

Let $\psi(x,y) \in L^1(\Omega;\mathcal{C}_{per}(Y))$. Then, for any positive value of ε, $\psi\left(x, \frac{x}{\varepsilon}\right)$ is a measurable function on Ω such that

$$\left|\left| \psi\left(x, \frac{x}{\varepsilon}\right) \right|\right|_{L^1(\Omega)} \leq ||\psi(x,y)||_{L^1(\Omega;\mathcal{C}_{per}(Y))} := \int_\Omega \sup_{y\in Y} |\psi(x,y)|\, dx.$$

3.1. Preliminary lemmas

First of all, let us recall the following result, which is the basis of all our analysis:

Lemma 1. *Let $b \in L^\infty(Y)$, $v_0 \in L^2(Y)$ with $\langle v_0 \rangle = 0$ and let $v = v(\tau, y) \in L^2_{loc}(0,\infty; H^1_{per}(Y))$ be the solution of*

$$\begin{cases} \partial_\tau v + \mathrm{div}_y(bv) - \Delta_y v = 0, \\ v(\tau = 0) = v_0. \end{cases} \tag{27}$$

Then there exist $\mu, C > 0$, depending only on N, Y and $||b||_{L^\infty}$, such that

$$||v(\tau)||_{L^\infty} \leq C||v_0||_{L^2} e^{-\mu\tau} \quad \forall \tau \geq 1,$$
$$||v(\tau)||_{L^2} \leq C||v_0||_{L^2} e^{-\mu\tau} \quad \forall \tau \geq 0.$$

Proof. This result is linked to the existence of gaps in the spectrum of the operator

$$Lw := -\Delta w + \mathrm{div}(bw).$$

We use the result stated in lemma 1.1 of [16] (see also the references therein). Let π be the invariant probability measure associated to the equation, i.e. π is the solution of

$$-\Delta_y \pi + \mathrm{div}_y(b\pi) = 0, \quad \langle \pi \rangle = 1.$$

$\pi \in H^1_{\mathrm{per}}(Y)$ exists and is unique, and there exists $\alpha > 0$ such that $\pi \geq \alpha$ by the Krein-Rutman theorem, and α depends only on Y, N and $||b||_\infty$.

Now, according to [16] we have for any $\mathcal{C}^2$ function $H : \mathbb{R} \to \mathbb{R}$

$$\partial_t \left[\pi H\left(\frac{v}{\pi}\right)\right] + \mathrm{div}_y \left[b\pi H\left(\frac{v}{\pi}\right)\right] - \frac{\partial}{\partial x_i}\left\{\pi^2 \frac{\partial}{\partial x_i}\left[\frac{1}{\pi} H\left(\frac{v}{\pi}\right)\right]\right\} = -\pi H''\left(\frac{v}{\pi}\right)\left|\nabla\left(\frac{v}{\pi}\right)\right|^2.$$

Take now $H(p) = \frac{1}{2}|p|^2$. Integrating this inequality on Y yields

$$\frac{1}{2}\frac{d}{dt}\int_Y \pi \left|\frac{v}{\pi}\right|^2 = -\int_Y \pi \left|\nabla\left(\frac{v}{\pi}\right)\right|^2.$$

But according to the Poincaré inequality for the measure π, there exists a constant $\nu > 0$ such that for any $w \in H^1_{\mathrm{per}}(Y)$,

$$\nu \int_Y \pi |w - \langle w \rangle_\pi|^2 \leq \int_Y \pi |\nabla w|^2,$$

where we have used the notation

$$\langle w \rangle_\pi := \frac{\int_Y w(y)\pi(y)\, dy}{\int_Y \pi(y)\, dy}.$$

(this inequality can be proved exactly along the same lines as the usual Poincaré inequality, for which $\pi \equiv 1$.) Hence

$$\frac{1}{2}\frac{d}{dt}\int_Y \pi \left|\frac{v}{\pi}\right|^2 \leq -\nu \int_Y \pi \left|\frac{v}{\pi} - \left\langle \frac{v}{\pi}\right\rangle_\pi\right|^2.$$

And since

$$\left\langle \frac{v}{\pi} \right\rangle_\pi = \frac{\int_Y v}{\int_Y \pi} = 0$$

we deduce that

$$\int_Y \pi \left|\frac{v}{\pi}\right|^2 \leq e^{-2\nu t} \int_Y \pi \left|\frac{v_0}{\pi}\right|^2$$

and we obtain the result announced in lemma 1. The exponential convergence in L^∞ norm follows from parabolic regularity.

Remark 2. For the reader's convenience, we provide here a short proof of the parabolic regularity result used here, for which we have failed to find an explicit reference. Assume that $u = u(\tau, y)$ is a solution of

$$\partial_\tau u + \mathrm{div}_y(bu) - \Delta_y u = 0, \tag{28}$$

where $b = b(\tau, y) \in L^\infty((0,\infty)\times Y)^N$ (we allow b to depend on time). Then there exists a constant $C > 0$, depending only on $||b||_{L^\infty}$, N and Y, such that for all $\tau \geq 0$,

$$||u(\tau+1)||_{L^\infty(Y)} \leq C||u(\tau)||_{L^1(Y)}. \tag{29}$$

Indeed, if u is a nonnegative solution of (28), then (29) follows from Harnack's inequality: there exists a constant C depending only on $||b||_{L^\infty}$, N and Y, such that for all $\tau \geq 0$

$$\max_{y\in Y} u(\tau+1, y) \leq C \min_{y\in Y} u(\tau+2, y).$$

Thus

$$||u(\tau+1)||_{L^\infty(Y)} \leq C\,\langle u(\tau+2)\rangle = C\,\langle u(\tau)\rangle = \frac{C}{|Y|}||u(\tau)||_{L^1(Y)}.$$

Hence (29) is proved for nonnegative solutions of (28). Now, if u is an arbitrary solution of (28) and $\tau_0 \geq 0$, we write $u(\tau = \tau_0) =: u_0 = (u_0)_+ - (u_0)_-$, with $a_+ := \max(a, 0)$ for any $a \in \mathbb{R}$, and $a_- = (-a)_+$. We denote by u^+, u^- the solutions of (28) for $\tau \geq \tau_0$ corresponding to initial data $u^+(\tau = \tau_0) = u_0^+$, $u^-(\tau = \tau_0) = u_0^-$. Then u^+ and u^- are nonnegative solutions of (28) and $u = u^+ - u^-$ for $\tau \geq \tau_0$. Thus

$$\max_Y u(\tau_0+1) \leq \max_Y u^+(\tau_0+1) \leq C\int_Y u^+(\tau_0),$$
$$\inf_Y u(\tau_0+1) \geq -\max_Y u^-(\tau_0+1) \geq -C\int_Y u^-(\tau_0)$$

and consequently

$$||u(\tau_0+1)||_{L^\infty} \leq C\int_Y (u_0^+ + u_0^-) \leq C\int_Y |u(\tau_0)|$$

which is the desired inequality for an arbitrary solution of (28).

We now generalize lemma 1 to the case when the coefficients might depend on the time variable t and on a parameter $x \in \mathbb{R}^N$. We wish to emphasize that the results stated in the two next lemmas are not optimal, but merely adapted to the problem addressed here.

Lemma 2. *Let $R > 0$.*

Let $b = b(\tau, x, y) \in L^\infty((0,\infty)\times B_R \times Y)$ such that $\mathrm{div}_y b \in L^\infty((0,\infty)\times B_R; L^2(Y))$, $v_0 = v_0(x, y) \in L^\infty(B_R; L^2(Y))$ with $\langle v_0(x,\cdot)\rangle = 0$ for almost every $x \in B_R$.

Let $v = v(\tau, x, y) \in L^\infty(B_R; L^2_{loc}(0,\infty; H^1_{per}(Y)))$ be the solution of

$$\begin{cases} \partial_\tau v + \mathrm{div}_y(bv) - \Delta_y v = 0, \\ v(\tau = 0) = v_0. \end{cases} \tag{30}$$

(x is a parameter of the above equation.)

Assume that there exists $b_\infty = b_\infty(x, y) \in L^\infty(B_R; L^\infty(Y))$ with $\mathrm{div}_y b_\infty \in L^\infty(B_R; L^2(Y))$ such that as $\tau \to \infty$

$$||b(\tau) - b_\infty||_{L^\infty(B_R; L^\infty(Y))} \to 0,$$
$$||\mathrm{div}_y b(\tau) - \mathrm{div}_y b_\infty||_{L^\infty(B_R; L^2(Y))} \to 0.$$

Then there exists a constant C depending only on N, Y and the bounds on b and b_∞ such that for all $\tau \geq 1$,

$$||v||_{L^\infty(B_R; L^2(\tau,\tau+1; H^2(Y)))} + ||v(\tau)||_{L^\infty(B_R; H^1(Y))} \leq C||v(\tau-1)||_{L^\infty(B_R; L^2(Y))}. \tag{31}$$

Moreover, there exists a constant C depending on N, Y, b and b_∞, and a constant $\mu > 0$ depending only on N, Y, and $||b_\infty||_{L^\infty(B_R\times Y)}$ such that

$$||v(\tau)||_{L^\infty(B_R;L^\infty(Y))} \leq C||v_0||_{L^\infty(B_R;L^2(Y))}e^{-\mu\tau} \quad \forall \tau \geq 1,$$
$$||v(\tau)||_{L^\infty(B_R;L^2(Y))} \leq C||v_0||_{L^\infty(B_R;L^2(Y))}e^{-\mu\tau} \quad \forall \tau \geq 0.$$

Proof. Let us treat x as a parameter. Let $L_\infty(x)$ be the differential operator

$$L_\infty(x)w(y) := -\Delta_y w(y) + \mathrm{div}_y(b_\infty(x,y)w(y));$$

then v satisfies the equation

$$\partial_\tau v + L_\infty(x)v = f(\tau,x,y),$$

where $f(\tau,x,y) = \mathrm{div}_y((b_\infty(x,y) - b(\tau,x,y))v(\tau,x,y))$.

According to standard regularity results on parabolic equations (see for instance [7]), we have on the one hand, for all $T > 0$ and for a.e. $x \in B_R$

$$\sup_{T\leq t\leq T+\frac{3}{2}} ||v(t,x)||_{H^1} + ||v(x)||_{L^2(T,T+\frac{3}{2};H^2(Y))} \leq C\left(||f(x)||_{L^2((T,T+\frac{3}{2})\times Y)} + ||v(T,x))||_{H^1}\right),$$

and on the other hand, for all $\tau \geq \frac{1}{2}$,

$$\sup_{\tau-\frac{1}{2}\leq t\leq \tau+\frac{1}{2}} ||v(t,x)||_{L^2(Y)} + ||v(x)||_{L^2(\tau-\frac{1}{2},\tau+\frac{1}{2};H^1(Y))} \leq C\left|\left|v\left(\tau-\frac{1}{2},x\right)\right|\right|_{L^2},$$

where C is a constant which depends only on N, Y and the bounds on b and b_∞.

For a.e. $x \in B_R$, we choose $T \in [\tau - \frac{1}{2}, \tau]$ such that

$$||v(T)||_{H^1} \leq \sqrt{2}||v(x)||_{L^2(\tau-\frac{1}{2},\tau;H^1(Y))}$$

and we evaluate

$$||f(x)||_{L^2((T,T+\frac{3}{2})\times Y)} \leq C\left[||v(x)||_{L^2(\tau-\frac{1}{2},\tau+\frac{3}{2};H^1)} + ||v(x)||_{L^\infty((\tau-\frac{1}{2},\tau+\frac{3}{2})\times Y)}\right].$$

Always thanks to parabolic regularity, there exists a constant C (depending only on N, Y and $||b||_{L^\infty}$) such that for all $\tau \geq 1$

$$||v(x)||_{L^\infty((\tau-\frac{1}{2},\tau+\frac{3}{2})\times Y)} \leq C||v(\tau-1,x)||_{L^2(Y)}.$$

Gathering all the terms, we obtain inequality (31) (notice that $[\tau,\tau+1] \subset [T, T+\frac{3}{2}]$).

Let us now prove the exponential convergence: let $U_\infty(\tau;x)$ be the evolution operator associated to the equation $\partial_\tau w + L_\infty(x)w = 0$, that is, $U_\infty(\tau;x)w_0 = w(\tau,y;x)$, where $w(x) \in L^2_{\mathrm{loc}}(0,\infty;H^1_{\mathrm{per}}(Y))$ is the solution of the system

$$\begin{cases} \partial_\tau w + \mathrm{div}_y(b_\infty w) - \Delta_y w = 0, \\ w(\tau=0) = w_0. \end{cases}$$

According to lemma 1, for all $w \in L^2(Y)$ such that $\langle w\rangle = 0$ and for almost every $x \in B_R$

$$||U_\infty(\tau;x)w||_{L^2} \leq C||w||_{L^2}e^{-\mu\tau} \quad \forall \tau \geq 0,$$

where C and μ are constants which depend only on N,Y, and the bounds on b_∞.

We use Duhamel's formula: for all $\tau > 0$, $x \in B_R$

$$v(\tau,x) = U_\infty(\tau;x)v_0 + \int_0^\tau U_\infty(\tau-\sigma;x)f(\sigma,x)\,d\sigma.$$

Notice that f has mean value zero; consequently,

$$||v(\tau,x)||_{L^2} \leq Ce^{-\mu\tau}||v_0(x)||_{L^2} + \int_0^\tau e^{-\mu(\tau-\sigma)}||f(\sigma,x)||_{L^2}\,d\sigma.$$

We bound $||f(\sigma,x)||_{L^2}$ by

$$\begin{aligned}&||b(\sigma)-b_\infty||_{L^\infty(Y)}||v(\sigma)||_{H^1}+||\mathrm{div}_y b(\sigma)-\mathrm{div}_y b_\infty||_{L^2(Y)}||v(\sigma)||_{L^\infty}\\ \leq\ & C\left(||b(\sigma)-b_\infty||_{L^\infty(Y)}+||\mathrm{div}_y b(\sigma)-\mathrm{div}_y b_\infty||_{L^2(Y)}\right)||v(\sigma-1)||_{L^2}.\end{aligned}$$

Let $\delta>0$ arbitrary. There exist $\sigma_\delta>0$ such that if $\sigma\geq\sigma_\delta$, then for a.e. $x\in B_R$

$$||b(\sigma,x)-b_\infty(x)||_{L^\infty(Y)}+||\mathrm{div}_y b(\sigma,x)-\mathrm{div}_y b_\infty(x)||_{L^2(Y)}\leq\delta.$$

Hence,

$$||v(\tau,x)||_{L^2}\leq Ce^{-\mu\tau}||v_0(x)||_{L^2}+C\int_0^{\sigma_\delta}e^{-\mu(\tau-\sigma)}||v(\sigma,x)||_{H^1}d\sigma+C\delta\int_{\sigma_\delta}^{\tau}e^{-\mu(\tau-\sigma)}||v(\sigma-1,x)||_{L^2}d\sigma.$$

Returning to equation (30), it is easily proved that

$$\int_0^{\sigma_\delta}||\nabla_y v(\sigma,x)||_{L^2}^2\,d\sigma\leq e^{||b||_\infty^2\sigma_\delta}||v_0(x)||_{L^2}^2.$$

(Other bounds are possible).

We are eventually led to

$$||v(\tau,x)||_{L^2}\leq C_\delta e^{-\mu\tau}||v_0(x)||_{L^2}+C\delta\int_{\sigma_\delta-1}^{\tau}e^{-\mu(\tau-\sigma)}||v(\sigma,x)||_{L^2}\,d\sigma,$$

where the constants C and C_δ depend on N, Y, b, b_∞, and the constant C_δ also depends on δ. We use Gronwall's lemma and we find

$$||v(\tau,x)||_{L^2}\leq C_\delta||v_0(x)||_{L^2}e^{-(\mu-C\delta)\tau}.$$

We take $\delta=\frac{\mu}{2C}$ and use once more parabolic regularity; since the inequality is true for almost every $x\in B_R$, the theorem is proved.

We now prove a third lemma which will be needed in the course of the proof.

Lemma 3. *Let $b=b(\tau,x,y)\in L^\infty_{loc}(\mathbb{R}^N;L^\infty((0,\infty)\times Y))$, $b_\infty=b_\infty(x,y)\in L^\infty_{loc}(\mathbb{R}^N;L^\infty(Y))$ two vector fields satisfying the same hypotheses as in lemma 2 for any $R>0$.*

Let $f=f(\tau,x,y)\in L^\infty_{loc}((0,\infty)\times\mathbb{R}^N,L^2(Y))$, and $v_0=v_0(x,y)\in L^\infty_{loc}(\mathbb{R}^N,L^\infty(Y))$. Assume that $\langle f(\tau,x,\cdot)\rangle=\langle v_0(x,\cdot)\rangle=0$ for almost every $(\tau,x)\in(0,\infty)\times\mathbb{R}^N$, and that for all $R>0$ there exists a constant $A_R>0$ such that

$$||f||_{L^\infty((0,\infty)\times B_R;L^2(Y))}\leq A_R. \tag{32}$$

Let $v=v(\tau,x,y)$ be the solution of

$$\begin{cases}\partial_\tau v(\tau,x,y)+\mathrm{div}_y(b(\tau,x,y)v(\tau,x,y))-\Delta_y v(\tau,x,y)=f(\tau,x,y),\\ v(\tau=0,x,y)=v_0(x,y).\end{cases}\tag{33}$$

Then for all $R>0$, there exists a constant C_R depending on N, Y, A_R, b and b_∞ such that

$$||v(\tau)||_{L^\infty(B_R;L^2(Y))}\leq C_R\left(1+||v_0||_{L^\infty(B_R,L^2(Y))}\right)\quad\forall\tau\geq0.$$

Moreover, if f can be written

$$f=\sum_{i=1}^N\partial_{y_i}f_i,$$

with $f_i\in L^\infty((0,+\infty)\times B_R\times Y)$ for all $R>0$ and for $1\leq i\leq N$, then for all $R>0$, there exists a constant C_R depending only on N, Y, $\max_{1\leq i\leq N}||f_i||_{L^\infty((0,+\infty)\times B_R\times Y)}$, b and b_∞ such that

$$||v(\tau)||_{L^\infty(B_R;L^\infty(Y))}\leq C_R\left(1+||v_0||_{L^\infty(B_R,L^2(Y))}\right)\quad\forall\tau\geq1, \tag{34}$$

$$||v(\tau)||_{L^\infty(B_R;L^\infty(Y))}\leq C_R\left(1+||v_0||_{L^\infty(B_R,L^\infty(Y))}\right)\quad\forall\tau\leq1. \tag{35}$$

Proof. Let $U(\tau,\sigma;x)$ be the evolution operator associated to the equation

$$\partial_\tau w + \mathrm{div}_y(bw) - \Delta_y w = 0$$

(x is still treated as a parameter). In other words, for any $\tau \geq \sigma \geq 0$, $\varphi \in L^2(Y)$, $w(\tau,x,y) := U(\tau,\sigma;x)\varphi$ satisfies

$$\begin{cases} \partial_\tau w + \mathrm{div}_y(b(\tau)w) - \Delta_y w = 0 & \text{for } \tau > \sigma, \\ w(\tau=\sigma,x,y) = \varphi(y). \end{cases}$$

In lemma 2, we have proved that for every $R > 0$, there exists $C_R, \mu_R > 0$ such that if $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; L^2(Y))$ satisfies $\langle \varphi(x,\cdot)\rangle = 0$ for almost every x, then

$$||U(\tau,\sigma;x)\varphi||_{L^\infty(B_R;L^\infty(Y))} \leq C_R ||\varphi||_{L^\infty(B_R;L^2(Y))} e^{-\mu_R(\tau-\sigma)} \quad \forall\, \tau \geq \sigma + 1 \geq 1,$$
$$||U(\tau,\sigma;x)\varphi||_{L^\infty(B_R;L^2(Y))} \leq C_R ||\varphi||_{L^\infty(B_R;L^2(Y))} e^{-\mu_R(\tau-\sigma)} \quad \forall\, \tau \geq \sigma \geq 0.$$

And by Duhamel's formula, we also have

$$v(\tau,x,y) = U(\tau,0;x)v_0(x,y) + \int_0^\tau U(\tau,\sigma;x) f(\sigma,x,y)\, d\sigma.$$

Thus, for any $R > 0$, for all $\tau \geq 0$,

$$\begin{aligned} ||v(\tau)||_{L^\infty(B_R;L^2(Y))} &\leq C_R e^{-\mu_R \tau} ||v_0||_{L^\infty(B_R;L^2(Y))} + C_R \int_0^\tau e^{-\mu_R(\tau-\sigma)}\, d\sigma \\ &\leq C_R \left(||v_0||_{L^\infty(B_R;L^2(Y))} + 1 \right). \end{aligned}$$

Inequalities (34) and (35) are direct consequences of theorems 8.1 and 7.1 in chapter III of [11].

3.2. Homogenization

We now apply the lemmas of the preceding subsection to the function $r(\tau,x,y) = w(\tau,x,y) - v(y,\bar{u}_0(x))$ and its derivatives. The proof is divided into six steps: in the first step, we restrict ourselves to smooth initial datas. In the second step, we prove that r converges towards 0 in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N; L^\infty(Y))$. In the third step, we prove that r converges to 0 exponentially fast, thanks to the second step and lemma 2. In the fourth step, we focus on what happens for small times : precisely, we derive a bound on $w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - u^\varepsilon(\alpha,x)$. This step is quite long because bounds on the derivatives of r are required. The fifth step is also concerned with the behavior of the solutions of (1) and (12) for small times, but is much shorter and less involved. Eventually, in the sixth and last step, we gather all the bounds derived in the previous steps and we prove the convergence result announced in theorem 2.

3.2.1. Restriction to smooth initial datas As often in the study of conservation laws, or more generally, of evolution problems which admit a contraction principle, it is enough to prove the result for smooth initial datas: indeed, choose, by density, a family of functions u_0^δ such that $u_0^\delta \in \mathcal{C}^\infty_{\mathrm{per}}(\mathbb{R}^N \times Y) \cap L^1_{\mathrm{loc}}(\mathbb{R}^N; \mathcal{C}_{\mathrm{per}}(Y))$, with the following properties:

$$\left|\left|u_0 - u_0^\delta\right|\right|_{L^1(B_R;\mathcal{C}_{\mathrm{per}}(Y))} \to 0 \quad \text{as } \delta \to 0 \quad \forall R > 0,$$
$$v(y, A-1) \leq u_0^\delta(x,y) \leq v(y, B+1) \quad \forall \delta > 0\ \forall (x,y) \in \mathbb{R}^N \times Y.$$

We denote by $u_\delta^\varepsilon(t,x)$, $v_\delta^\varepsilon(t,x)$ the solutions of (1) corresponding to initial datas $u_0^\delta\left(x, \frac{x}{\varepsilon}\right)$, $v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)$ respectively, and $\bar{u}_0^\delta(x) = \left\langle u_0^\delta(x,\cdot)\right\rangle$. Then there exists a constant $K' \geq K$ depending only on N, Y, A, B, m, n and C_0 such that

$$|u_\delta^\varepsilon(t,x)| + |v_\delta^\varepsilon(t,x)| \leq K' \quad \forall (\tau,x,y) \in [0,\infty) \times \mathbb{R}^N \times Y,\ \forall \varepsilon, \delta > 0$$

and

$$-K' \leq v(y,p) \leq K' \quad \forall y \in Y\ \forall p \in [A-1, B+1].$$

Hence the following inequalities hold, according to the L^1 contraction principle for equation (1)

$$||u^\varepsilon(T)-u^\varepsilon_\delta(T)||_{L^1(B_R)}\le e^{\frac{CT}{R}}\left(e^{-CR}+||u_0-u_0^\delta||_{L^1(B_R;\mathcal{C}_{\text{per}}(Y))}\right) \tag{36}$$

$$||v^\varepsilon(T)-v^\varepsilon_\delta(T)||_{L^1(B_R)}\le e^{\frac{CT}{R}}\left(e^{-CR}+||v(y,\bar u_0(x))-v(y,\bar u_0^\delta(x))||_{L^1(B_{2R};\mathcal{C}_{\text{per}}(Y))}\right) \tag{37}$$

for all $R>0$, $T>0$, and for some constant $C>0$ depending only on $||a||_{L^\infty(Y\times(-K',K'))}$.

Assume that we can prove that

$$u^\varepsilon_\delta - v^\varepsilon_\delta$$

goes to 0 in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^N)$ as $\varepsilon\to 0$, for all $\delta>0$. Then for any $0<a<b$, $R>1$

$$\begin{aligned}||u^\varepsilon-v^\varepsilon||_{L^1((a,b)\times B_R)}\le C\big(&||u^\varepsilon_\delta-v^\varepsilon_\delta||_{L^1((a,b)\times B_R)}+e^{-CR}\\ &+||u_0-u_0^\delta||_{L^1(B_R;\mathcal{C}_{\text{per}}(Y))}+||v(y,\bar u_0(x))-v(y,\bar u_0^\delta(x))||_{L^1(B_R;\mathcal{C}_{\text{per}}(Y))}\big).\end{aligned}$$

Let $\eta>0$ arbitrary, $0<a<b$ fixed. Take $R>0$ large enough so that $Ce^{-CR}\le\eta$. For this $R>0$, we now choose $\delta>0$ such that

$$C\left(||u_0-u_0^\delta||_{L^1(B_R;\mathcal{C}_{\text{per}}(Y))}+||v(y,\bar u_0(x))-v(y,\bar u_0^\delta(x))||_{L^1(B_R;\mathcal{C}_{\text{per}}(Y))}\right)\le\eta.$$

This is always possible since the second term of the left-hand side is bounded by

$$\left|\left|\frac{\partial v}{\partial p}\right|\right|_{L^\infty(Y\times[A-1,B+1])}||\bar u_0-\bar u_0^\delta||_{L^1(B_R)}\le C||u_0-u_0^\delta||_{L^1(B_R;\mathcal{C}_{\text{per}}(Y))}$$

where C is a constant which depends only on N, Y, and $||a||_{W^{1,\infty}(Y\times(-K',K'))}$.

Now, for this choice of R and δ, we take $\varepsilon_0>0$ small enough so that for all $\varepsilon\le\varepsilon_0$

$$||u^\varepsilon_\delta-v^\varepsilon_\delta||_{L^1((a,b)\times B_R)}\le\eta.$$

Hence, for all $\eta>0$, $0<a<b$, $R>0$, we have found $\varepsilon_0>0$ so that for all $\varepsilon\le\varepsilon_0$

$$||u^\varepsilon-v^\varepsilon||_{L^1((a,b)\times B_R)}\le 3\eta.$$

This is exactly saying that $u^\varepsilon-v^\varepsilon$ goes to 0 in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^N)$.

Thus, we now restrict ourselves to initial datas which have as much regularity as desired (this hypothesis will be made precise in the course of the proof). We work with $\delta>0$ fixed, and thus we drop all δ's, and we write K instead of K'.

3.2.2. Convergence of r in L^∞ Let us prove that

$$||r(\tau)||_{L^\infty_{\text{loc}}(\mathbb{R}^N;L^\infty(Y))}\to 0\quad\text{as }\tau\to\infty. \tag{38}$$

Thanks to section 2, we already know that for almost every $x\in\mathbb{R}^N$, $||r(\tau,x)||_{L^\infty(Y)}\to 0$ as $\tau\to\infty$. We deduce easily that

$$||r(\tau)||_{L^p_{\text{loc}}(\mathbb{R}^N;L^\infty(Y))}\to 0\quad\text{as }\tau\to\infty$$

for all $p\in[1,\infty)$. However, we need to prove that the same convergence holds with $p=\infty$: indeed, if we try and prove lemma 2 with the hypotheses on b replaced by

$$||b(\tau)-b_\infty||_{L^p_{\text{loc}}(\mathbb{R}^N,L^2(Y))}\to 0$$

for some $p<\infty$ (idem with $\text{div}b-\text{div}b_\infty$), then in the course of the proof, we are led to an inequality of the type

$$\begin{aligned}||v(\tau)||_{L^q(B_R;L^1(Y))}\le{}& Ce^{-\mu\tau}||v_0||_{L^q(B_R;L^1(Y))}\\ &+C\int_0^\tau e^{-\mu(\tau-\sigma)}||b(\sigma)-b_\infty||_{L^p(B_R,L^2(Y))}||v(\sigma-1)||_{L^r(B_R;L^1(Y))}\,d\sigma\\ &+C\int_0^\tau e^{-\mu(\tau-\sigma)}||\text{div}b(\sigma)-\text{div}b_\infty||_{L^p(B_R,L^2(Y))}||v(\sigma-1)||_{L^r(B_R;L^1(Y))}\,d\sigma\end{aligned}$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. If we cannot take $p = \infty$, then $r > q$ and we can no longer apply Gronwall's lemma.

In order to prove (38), we go back to the proof of theorem (1) and we define the quantities $U(t, x, y)$, $p^*(t, x)$, $p^*(x)$ (x is a parameter of the equation).

If we look closely at the proof, we see that it is enough to prove that $p^*(t, x)$ converges to $p^*(x) = \bar{u}_0(x)$ locally uniformly in x as $t \to \infty$. Since the functions $p^*(t, x)$ are decreasing w.r.t. $t > 0$, and $\bar{u}_0(x)$ is continuous in x if $u_0(x, y)$ is continuous in x uniformly in $y \in Y$, according to Dini's theorem we only have to prove that $p^*(t, x)$ is continuous w.r.t. x for every $t > 0$.

Let $\delta > 0$, $R > 0$. We assume that $u_0 \in \mathcal{C}_{\text{per}}(\mathbb{R}^N \times Y)$. There exists $\eta > 0$ such that

$$\forall x, x' \in B_R \quad |x - x'| \leq \eta \Rightarrow ||u_0(x, \cdot) - u_0(x', \cdot)||_{L^1(Y)} \leq \delta.$$

Let $x \in B_R$ such that $|x - x'| \leq \eta$. By the L^1 contraction property,

$$||u(\tau, x) - u(\tau, x')||_{L^1(Y)} \leq \delta \quad \forall \tau \geq 0,$$

which entails, thanks to parabolic regularity results (see remark 2)

$$||u(\tau, x) - u(\tau, x')||_{L^\infty(Y)} \leq C\delta \quad \forall \tau \geq 1$$

for some constant C depending only on N,Y, R and $||a||_{L^\infty(Y \times (-K,K))}$.

From this we deduce easily that

$$||U(\tau, x) - U(\tau, x')||_{L^\infty(Y)} \leq C\delta \quad \forall \tau \geq 1.$$

Let $p < p^*(\tau, x')$ arbitrary, $\tau \geq 1$. There exists $y \in Y$ such that

$$v(y, p) < U(\tau, x', y) \leq U(\tau, x, y) + C\delta \leq v(y, p^*(\tau, x)) + C\delta.$$

Hence, either $p \leq p^*(\tau, x)$ or

$$V(y) := v(y, p) - v(y, p^*(\tau, x)) \geq 0 \quad \forall y \in Y,$$
$$\inf_{y \in Y} V(y) \leq C\delta$$

and V satisfies an elliptic equation of the type

$$-\Delta_y V + \text{div}_y(bV) = 0,$$

with $b \in L^\infty(Y)$, $||b||_{L^\infty(Y)} \leq ||a||_{L^\infty(Y \times (-K,K))}$.

In the second case, according to Harnack's inequality (see [8]) there exists a constant C depending only on N,Y, and $||a||_{L^\infty(Y \times (-K,K))}$, such that $\sup_Y V \leq C \inf_Y V \leq C\delta$, and in that case

$$|p - p^*(\tau, x)| = \frac{1}{|Y|} ||v(\cdot, p) - v(\cdot, p^*(\tau, x))||_{L^1(Y)} \leq C\delta.$$

In all cases,

$$p < p^*(\tau, x') \Rightarrow p \leq p^*(\tau, x) + C\delta$$

and thus $p^*(\tau, x') \leq p^*(\tau, x) + C\delta$. Of course, the inequality obtained by exchanging the roles of x and x' holds and $p^*(\tau)$ is thus continuous in x for all $\tau \geq 1$.

3.2.3. r **converges exponentially fast to 0** $r(\tau, x, y)$ satisfies an equation of the type (30), with

$$b(\tau, x, y) := \int_0^1 a(y, v(y, \bar{u}_0(x)) + sr(\tau, x, y))\, ds.$$

Consequently, setting

$$b_\infty(x, y) := a(y, v(y, \bar{u}_0(x))),$$

we have

$$||b(\tau) - b_\infty||_{L^\infty(B_R; L^\infty(Y))} \leq C_1 ||r(\tau)||_{L^\infty(B_R; L^\infty(Y))},$$
$$||\mathrm{div}_y b(\tau) - \mathrm{div}_y b_\infty||_{L^\infty(B_R; L^2(Y))} \leq C_1 \left(1 + ||r(\tau)||_{L^\infty(B_R; H^1_{\mathrm{per}}(Y))}\right),$$

where

$$C_1 := ||\partial_v a||_{L^\infty(Y \times (-K,K))^N} + ||\mathrm{div}_y a||_{L^\infty(Y \times (-K,K))}.$$

Notice that in lemma 2, inequality (31) is established without using the assumption that $||\mathrm{div}_y b(\tau) - \mathrm{div}_y b_\infty|| \to 0$; in fact, we only need to prove that

$$||\mathrm{div}_y b - \mathrm{div}_y b_\infty||_{L^\infty(B_R; L^2((\tau - \frac{1}{2}, \tau + \frac{3}{2}) \times Y))}$$

is bounded (uniformly in $\tau \geq \frac{1}{2}$), and thus that

$$||r||_{L^\infty(B_R; L^2(\tau - \frac{1}{2}, \tau + \frac{3}{2}; H^1_{\mathrm{per}}(Y)))}$$

is bounded uniformly in τ. But we have

$$||r||_{L^\infty(B_R; L^2(\tau - \frac{1}{2}, \tau + \frac{3}{2}; H^1_{\mathrm{per}}(Y)))} \leq C ||r(\tau - \frac{1}{2})||_{L^\infty(B_R; L^2(Y))},$$

where C is a constant which depends only on N, Y, and $||b||_{L^\infty(0,\infty) \times B_R \times Y} \leq ||a||_{L^\infty(Y \times (-K,K))^N}$. Hence inequality (31) is satisfied for r. And the continuity of $\mathrm{div}_y a$ and $\partial_v a$ entails that $\mathrm{div} b(\tau) - \mathrm{div} b_\infty$ converges to 0 in $L^\infty(B_R; L^2(Y))$ as $\tau \to \infty$.

Consequently the hypotheses of lemma 2 are satisfied and

$$||r(\tau)||_{L^\infty(B_R; L^\infty(Y))} \leq C_R e^{-\mu \tau} ||r_0||_{L^\infty(B_R; L^2(Y))} \quad \forall \tau \geq 1 \tag{39}$$

where $\mu > 0$ is a constant which depends only on N, Y, and $||a||_{L^\infty(Y \times (-K,K))}$

3.2.4. Behavior of $u^\varepsilon(t, x)$ for small times In this paragraph, we derive a bound on $w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - u^\varepsilon(\alpha, x)$.

Assume that the initial data $u_0(x, y)$ is smooth in x and y, namely

$$\nabla_x u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathcal{C}^1_{\mathrm{per}}(Y))^N \quad \text{and } \Delta_x u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathcal{C}_{\mathrm{per}}(Y)).$$

Then $\nabla_x w(\tau, x, y) \in L^\infty_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^N; \mathcal{C}^1_{\mathrm{per}}(Y))^N$, $\Delta_x u_0 \in L^\infty_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^N; \mathcal{C}_{\mathrm{per}}(Y))$.

Let us compute

$$\frac{\partial}{\partial t} w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right) - \varepsilon \Delta_x w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right)$$
$$= \sum_{i=1}^N \frac{\partial w}{\partial x_i}\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) a_i\left(\frac{x}{\varepsilon}, w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right) - \Delta_{xy} w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - \varepsilon \Delta_x \left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) =: f_\varepsilon(t, x)$$

and

$$||f_\varepsilon(t)||_{L^1(B_R)} \leq C\left(\left|\left|\nabla_x w\left(\frac{t}{\varepsilon}\right)\right|\right|_{L^1(B_R; W^{1,\infty}(Y))} + ||\Delta_x w||_{L^1(B_R; L^\infty(Y))}\right)$$

where the constant C depends only on $||a||_{L^\infty(Y \times [-K,K])}$. Thus, we have to prove that $\nabla_x w(\tau, x, y)$ (resp. $\Delta_x w(\tau, x, y)$) is bounded in $L^1(B_R; W^{1,\infty}(Y))$ (resp. $L^1(B_R; L^\infty(Y))$) uniformly in τ.

With this aim in view, we define

$$R_i(\tau,x,y) := \frac{\partial r}{\partial x_i}(\tau,x,y) = \frac{\partial w}{\partial x_i}(\tau,x,y) - \frac{\partial \bar{u}_0(x)}{\partial x_i} m(y,\bar{u}_0(x)),$$

where $m(\cdot,p)\in H^1_{\text{per}}(Y)$ is the solution of

$$-\Delta_y m(y,p) + \text{div}_y(a(y,v(y,p))m(y,p)) = 0.$$

Thanks to elliptic regularity results and the regularity hypotheses on a, $m(\cdot,p)\in W^{2,q}_{\text{per}}(Y)$ for all $p\in\mathbb{R}$ and for all $1\leq q<\infty$, and for all $R>0$ there exists a constant $C_R>0$ depending only on N, Y, a and R such that

$$||m(\cdot,p)||_{L^\infty(Y)} + ||m(\cdot,p)||_{H^1(Y)} \leq C_R \quad \forall |p|\leq R.$$

R_i satisfies

$$\partial_\tau R_i + \text{div}_y(c(\tau,x,y)R_i) - \Delta_y R_i = \text{div}_y\left((b_\infty(x,y) - c(\tau,x,y))\frac{\partial \bar{u}_0(x)}{\partial x_i} m(y,\bar{u}_0(x))\right) =: f_i(\tau,x,y)$$

where

$$c(\tau,x,y) := a(y,w(\tau,x,y)),$$
$$b_\infty(x,y) = a(y,v(y,\bar{u}_0(x))).$$

As a direct application of lemma 3, we deduce that R_i is bounded in $L^\infty([0,\infty)\times B_R;\mathcal{C}_{\text{per}}(Y))$ for all $R>0$. Indeed we only have to check that f_i is bounded in $L^\infty((0,\infty)\times B_R;L^2(Y))$; since

$$||\text{div}_y c(\tau) - \text{div}_y b_\infty||_{L^\infty(B_R;L^2(Y))} \leq C\left(1 + ||\nabla_y r(\tau)||_{L^\infty(B_R;L^2(Y))} + \sup_{p\in[-K,K]} ||\nabla_y v(y,p)||_{L^2(Y)}\right),$$

we thus obtain a bound on f_i thanks to the first step, inequality (31) and proposition 1. Hence $\nabla_x w(\tau,x,y)$ is bounded in $L^\infty(B_R,\mathcal{C}_{\text{per}}(Y))$ uniformly in τ.

In order to prove that the y derivatives are bounded as well, we intend to use theorem 11.1 in chapter III of [11]; hence we have to prove that

$$\frac{\partial c_j}{\partial y_i} \in L^{2r}(T,T+1;L^{2q}(Y)),$$

with uniform bounds in $T>0$, $x\in B_R$, for some $q\geq 1, r\geq 1$ such that

$$\frac{1}{r} + \frac{N}{2q} = 1-\kappa, \tag{40}$$

with $\kappa\in(0,1)$ if $N\geq 2$ and $\kappa\in(0,\frac{1}{2})$ if $N=1$. Consequently we must prove that

$$\nabla_y w(\tau,x,y) \in L^{2r}(T,T+1;L^{2q}(Y))$$

with uniform bounds in $T>0$ and $x\in B_R$. However at the moment we only know that

$$\nabla_y w(\tau,x,y) \in L^\infty((0,\infty)\times B_R;L^2(Y)) \cap L^\infty(B_R;L^2_{\text{loc}}((0,\infty);H^1_{\text{per}}(Y)))^N$$

which is not sufficient to ensure that w has the desired regularity when N is large. Hence we first need to prove the

Lemma 4. *There exist q,r satisfying* (40), *and a constant C_R depending only on N, Y, R, K and $||a||_{W^{1,\infty}(Y\times(-K,K))}$ such that for all $T>0$*

$$||\nabla_y w||_{L^\infty(B_R;L^{2r}(T,T+1;L^{2q}(Y)))} \leq C_R.$$

Proof. Let

$$S_i(\tau,x,y) := \frac{\partial r(\tau,x,y)}{\partial y_i}.$$

Then S_i satisfies

$$\partial_\tau S_i + \operatorname{div}(cS_i) - \Delta_y S_i = \operatorname{div}(F_i),$$

where

$$F_i(\tau,x,y) := -\frac{\partial c(\tau,x,y)}{\partial y_i} r(\tau,x,y).$$

Then

$$|F_i(\tau,x,y)| \le C(1+|S_i(\tau,x,y)|),$$

where the constant C depends only on $||\partial_v a||_{L^\infty(Y\times(-K,K))}$, $||\partial_{y_i} a||_{L^\infty(Y\times(-K,K))}$, and K. Moreover,

$$||c||_{L^\infty([0,\infty)\times\mathbb{R}^N\times Y)} \le ||a||_{L^\infty(Y\times(-K,K))}.$$

Thus, according to theorem 9.1 and corollary 9.2 in chapter III of [11], we have

$$\begin{cases} S_i \in L^\infty(B_R; L^{2r}(T,T+1; L^{2q}(Y))) \\ \text{with } \dfrac{1}{r} + \dfrac{N}{2q} = 1 + \dfrac{N\theta}{2},\ \theta\in(0,1) \end{cases} \Rightarrow \begin{cases} S_i \in L^\infty(B_R; L^{2r'}(T,T+1; L^{2q'}(Y))) \\ \text{with } \dfrac{1}{r'} + \dfrac{N}{2q'} = \dfrac{N\theta}{2} \end{cases} \tag{41}$$

and $||S_i||_{L^\infty(B_R;L^{2r'}(T,T+1;L^{2q'}(Y)))}$ is bounded by a constant which depends only on N, Y, $||a||_{W^{1,\infty}(Y\times(-K,K))}$, θ and $||S_i||_{L^\infty(B_R;L^{2r}(T,T+1;L^{2q}(Y)))}$.

Moreover, by interpolation we know that

$$\begin{aligned} ||S_i||_{L^\infty(B_R;L^{2r}(T,T+1;L^{2q}(Y)))} &\le ||S_i||^{1-\theta}_{L^\infty(B_R;L^\infty(T,T+1;L^2(Y)))} ||S_i||^{\theta}_{L^\infty(B_R;L^2(T,T+1;L^{q_0}(Y)))} \\ &\le C||S_i||^{1-\theta}_{L^\infty(B_R;L^\infty(T,T+1;L^2(Y)))} ||S_i||^{\theta}_{L^\infty(B_R;L^2(T,T+1;H^1(Y)))} \end{aligned}$$

where the constant C depends only on N and Y and where

$$\frac{1}{q_0} = \frac{1}{2} - \frac{1}{N},$$
$$\theta = \frac{1}{r} \in [0,1],$$
$$\frac{1}{2q} = \frac{\theta}{q_0} + \frac{1-\theta}{2}.$$

Hence we have $S_i \in L^\infty(B_R; L^{2r_1}(T,T+1; L^{2q_1}(Y)))$ for all $(q_1,r_1)\in[1,\infty)$ such that

$$\frac{N}{2q_1} + \frac{1}{r_1} = \frac{N}{2} = 1 + \frac{N\theta_1}{2},$$

where $\theta_1 = 1 - \frac{2}{N}$ (notice that in the case when $N=1$, we need not go further). Define the sequence $(\theta_k)_{k\ge0}$ in $\mathbb{R}$ by

$$1 + \frac{N\theta_{k+1}}{2} = \frac{N\theta_k}{2}.$$

Then it is easily proved by induction that as long as $\theta_k\in(0,1)$,

$$\begin{cases} S_i \in L^\infty(B_R; L^{2r}(T,T+1; L^{2q}(Y))) \\ \forall (q,r)\in[1,\infty)^2 \text{ s.t. } \exists\, \theta\in(\theta_k,1),\ \dfrac{1}{r} + \dfrac{N}{2q} = \dfrac{N\theta}{2}. \end{cases}$$

And the bound on $||S_i||_{L^\infty(B_R;L^{2r}(T,T+1;L^{2q}(Y)))}$ depends only on N, Y, K, $||a||_{W^{1,\infty}(Y\times(-K,K))}$ and θ. In particular, it is uniform in T,R.

Since $\theta_k = 1 - k\frac{2}{N}$, we can choose k_0 such that $\theta_k\in(0,1)$ for $k\le k_0$ and $0 \le \frac{N\theta_k}{2} < 1$. Then we choose $\theta\in(\theta_k,1)$ such that $0 < \frac{N\theta}{2} < 1$, and $q,r\ge1$ such that $\frac{1}{r} + \frac{N}{2q} = \frac{N\theta}{2}$. According to the above remarks, $S_i\in L^\infty(B_R; L^{2r}(T,T+1; L^{2q}(Y)))$ with uniform bounds in $T>0$. Since

$$\nabla_y w = (S_1,\cdots,S_N) + \nabla_y v(y,\bar u_0(x)),$$

this concludes the proof of the lemma.

Consequently, according to theorem 11.1 in chapter III of [11], there exists a constant C_R depending only on $R > 0$, N, Y, K, $||a||_{W^{1,\infty}(Y\times(-K,K))}$ and $||\nabla_x u_0||_{L^\infty(B_R,\mathcal{C}_{\rm per}(Y))}$ such that

$$||\partial^2_{y_i x_j} w||_{L^\infty([1,\infty)\times B_R\times Y)} \le C_R \quad \forall\, 1\le i,j\le N.$$

And for every $\tau\in[0,1]$,

$$||\partial^2_{y_i x_j} w(\tau)||_{L^\infty(B_R\times Y)} \le C_R\left(1+||u_0||_{W^{1,\infty}(B_R;\mathcal{C}^1_{\rm per}(Y))}\right).$$

It remains to prove that $\Delta_x w$ is bounded in $L^\infty([0,\infty)\times B_R\times Y)$. The equation satisfied by $R(\tau,x,y) := \Delta_x w(\tau,x,y)$ is

$$\partial_\tau R + {\rm div}_y(cR) - \Delta_y R = \frac{\partial f_i}{\partial x_i} - {\rm div}_y((\partial_{x_i} c)R);$$

the right-hand side of the above equation belongs to $L^\infty([0,\infty)\times B_R\times Y)$ for all $R>0$ according to the preceding steps. Thus $\Delta_x w$ is bounded in $L^\infty([0,\infty)\times B_R\times Y)$ by lemma 3.

Now, we multiply the L^1 contraction principle inequality between u^ε and $w\left(\frac{t}{e},x,\frac{x}{\varepsilon}\right)$ by a test function $\varphi\left(\frac{x}{R}\right)$, with $\varphi\in\mathcal{D}(\mathbb{R}^N)$, $0\le\varphi\le 1$, $\varphi(x)=1$ when $|x|\le 1$, $\varphi(x)=0$ when $|x|\ge 2$, and we integrate over $(0,\alpha)\times\mathbb{R}^N$.

We deduce there exist constants C, C_R such that C depends only on N and $||a||_{L^\infty(Y\times[-K,K])}$ and C_R depends on N, Y, K, R, $||a||_{W^{1,\infty}(Y\times(-K,K))}$, $||u_0||_{W^{1,\infty}(B_{2R};\mathcal{C}^1_{\rm per}(Y))}$ and $||\Delta_x u_0||_{L^\infty(B_{2R};\mathcal{C}_{\rm per}(Y))}$ such that

$$\begin{aligned}\left|\left|u^\varepsilon(\alpha) - w\left(\frac{\alpha}{\varepsilon},x,\frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_R)} \le\ & \underbrace{||u^\varepsilon(t=0) - w\left(\tau=0,x,\frac{x}{\varepsilon}\right)||_{L^1(B_{2R})}}_{=0}\\ & +\frac{C}{R}\int_0^\alpha \left|\left|u^\varepsilon(s) - w\left(\frac{s}{\varepsilon},x,\frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_{2R})} ds\\ & + C_R\alpha.\end{aligned}$$

For $s>0$,

$$\begin{aligned}\left|\left|u^\varepsilon(s) - w\left(\frac{s}{\varepsilon},x,\frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_{2R})} &\le ||u^\varepsilon(s)||_{L^1(B_{2R})} + \left|\left|w\left(\frac{s}{\varepsilon},x,\frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_{2R})}\\ &\le 2\,|B_{2R}|\,K.\end{aligned}$$

Thus

$$\left|\left|u^\varepsilon(\alpha) - w\left(\frac{\alpha}{\varepsilon},x,\frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_R)} \le C_R\alpha$$

where the constant C_R depends only on R, N, Y, K, $||a||_{W^{1,\infty}(Y\times(-K,K))}$, $||u_0||_{W^{1,\infty}(B_{2R};\mathcal{C}^1_{\rm per}(Y))}$ and $||\Delta_x u_0||_{L^\infty(B_{2R};\mathcal{C}_{\rm per}(Y))}$.

3.2.5. Bound on $\left|\left|v\left(\frac{x}{\varepsilon},\bar u(\alpha,x)\right) - v\left(\frac{x}{\varepsilon},\bar u_0(x)\right)\right|\right|_{L^1(B_R)} + \left|\left|v^\varepsilon(\alpha) - v\left(\frac{x}{\varepsilon},\bar u(\alpha,x)\right)\right|\right|_{L^1(B_R)}$

– First, $A\le \bar u_0(x)\le B$ a.e. in $\mathbb{R}^N$. Hence, $A\le\bar u(t,x)\le B$ for a.e. $(t,x)\in(0,\infty)\times\mathbb{R}^N$. Thus

$$\begin{aligned}\left|\left|v\left(\frac{x}{\varepsilon},\bar u(\alpha,x)\right) - v\left(\frac{x}{\varepsilon},\bar u_0(x)\right)\right|\right|_{L^1(B_R)} &\le \left|\left|\frac{\partial v(y,p)}{\partial p}\right|\right|_{L^\infty(Y\times[A,B])} ||\bar u(\alpha) - \bar u_0(x)||_{L^1(B_R)}\\ &\le \left|\left|\frac{\partial v(y,p)}{\partial p}\right|\right|_{L^\infty(Y\times[A,B])}\omega(\alpha) \qquad (42)\end{aligned}$$

where ω is the modulus of continuity in time for $\bar u$:

$$\omega(k) := \sup_{0\le\tau\le k} ||\bar u(\tau) - \bar u_0||_{L^1(\mathbb{R}^N)} \underset{k\to 0}{\longrightarrow} 0$$

– For all $R>0$ and for almost every $\alpha>0$,

$$\left|\left|v^\varepsilon(\alpha) - v\left(\frac{x}{\varepsilon},\bar u(\alpha,x)\right)\right|\right|_{L^1(B_R)} \to 0 \qquad (43)$$

as $\varepsilon\to 0$ according to the homogenization result in case of well-prepared initial data.

3.2.6. Conclusion Gathering (42), (43), and the exponential result proved in the second step, we obtain, for all $R > 0$

$$||u^\varepsilon(\alpha) - v^\varepsilon(\alpha)||_{L^1(B_R)} \le C_R\alpha + C\omega(\alpha) + \left|\left|v^\varepsilon(\alpha,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)} + C'_R e^{-\frac{\mu\alpha}{\varepsilon}},$$

as long as $\alpha \ge \varepsilon > 0$, where :

- the constant C_R depends on R, N, Y, K, $||a||_{W^{1,\infty}(Y\times(-K,K))}$, $||u_0||_{W^{1,\infty}(B_{2R};\mathcal{C}^1_{\text{per}}(Y))}$ and $||\Delta_x u_0||_{L^\infty(B_{2R};\mathcal{C}_{\text{per}}(Y))}$;
- the constants C, μ depend only on N, Y, and $||a||_{W^{1,\infty}(Y\times[-K',K'])}$;
- the constant C'_R depends on R, N, Y, K, $||a||_{W^{1,\infty}(Y\times[-K,K])}$, $||u_0(x,y)-v(y,\bar{u}_0(x))||_{L^\infty(B_R;L^2(Y))}$, and the speed of convergence of $w(\tau,x,y)$ towards $v(y,\bar{u}_0(x))$ in $L^\infty(B_R\times Y)$.

Going back to inequality (26), and using all the bounds derived in the preceding steps leads us to

$$||u^\varepsilon(T)-v^\varepsilon(T)||_{L^1(B_R)} \le Ce^{CT}\left[e^{-R} + C_R\alpha + \omega(\alpha) + \left|\left|v^\varepsilon(\alpha,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)} + C'_R e^{-\frac{\mu\alpha}{\varepsilon}}\right].$$

The inequality holds for all $R > 0$ and for all $T > \alpha \ge \varepsilon > 0$.

Let $\eta > 0$ arbitrary, $T > 0$ fixed. First, we choose $R > 0$ large enough so that

$$Ce^{CT}e^{-R} \le \eta.$$

For this $R > 0$, we choose $\alpha_0 > 0$ small enough so that

$$Ce^{CT}\left(C_R\alpha + C\omega(\alpha)\right) \le \eta \quad \text{for } 0 < \alpha \le \alpha_0.$$

We pick $\alpha \in (0,\alpha_0)$ such that

$$\left|\left|v^\varepsilon(\alpha,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)} \to 0 \quad \text{as } \varepsilon \to 0.$$

At last, for this R and $\alpha > 0$, we take $0 < \varepsilon_0 \le \alpha$ small enough so that

$$Ce^{CT}\left(\left|\left|v^\varepsilon(\alpha,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)} + C'_R e^{-\frac{\mu\alpha}{\varepsilon}}\right) \le \eta \quad \forall \varepsilon \in (0,\varepsilon_0].$$

Hence, we have proved that

$$||u^\varepsilon - v^\varepsilon||_{L^\infty_{\text{loc}}((0,\infty);L^1_{\text{loc}}(\mathbb{R}^N))} \to 0 \quad \text{as } \varepsilon \to 0.$$

Let $0 < a < b$; since

$$\begin{aligned}\left|\left|u^\varepsilon - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right|\right|_{L^1((a,b)\times B_R)} &\le (b-a)||u^\varepsilon - v^\varepsilon||_{L^\infty((a,b);L^1(B_R))} \\ &\quad + \left|\left|v^\varepsilon - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right|\right|_{L^1((a,b)\times B_R)}\end{aligned}$$

we conclude that

$$u^\varepsilon - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0 \quad \text{as } \varepsilon \to 0 \text{ in } L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^N).$$

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