Homogenization of a quasilinear parabolic equation with vanishing viscosity

Anne-Laure Dalibard

CEREMADE-UMR 7534, Université Paris-Dauphine, Place du maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, FRANCE

Abstract

We study the limit as $\varepsilon \to 0$ of the solutions of the equation $\partial_t u^\varepsilon + \text{div}_x \left[A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\right] - \varepsilon\Delta_x u^\varepsilon = 0$. After computing the homogenized problem thanks to formal double-scale expansions, we prove that as ε goes to 0, u^ε behaves in L^2_{loc} as $v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$, where v is determined by a cell problem and $\bar{u}$ is the solution of the homogenized problem. The proof relies on the use of two-scale Young measures, a generalization of Young measures adapted to two-scale homogenization problems.

Résumé

On étudie ici la limite quand $\varepsilon \to 0$ des solutions de l'équation $\partial_t u^\varepsilon + \text{div}_x \left[A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\right] - \varepsilon\Delta_x u^\varepsilon = 0$. Après avoir identifié le problème homogénéisé grâce à un développement asymptotique, on montre que u^ε se comporte dans L^2_{loc} comme $v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$ lorsque $\varepsilon \to 0$, où v est la solution d'un problème de la cellule et $\bar{u}$ celle du problème homogénéisé. La preuve utilise les mesures d'Young à deux échelles, une généralisation des mesures d'Young adaptée aux problèmes d'homogénéisation à deux échelles.

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1 Introduction

This paper is devoted to the analysis of the behavior as $\varepsilon \to 0$ of the solutions $u^\varepsilon \in L^\infty_{\text{loc}}([0,\infty) \times \mathbb{R}^N) \cap \mathcal{C}([0,\infty), L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^2_{\text{loc}}([0,\infty), H^1_{\text{loc}}(\mathbb{R}^N))$ of the parabolic

Email address: `dalibard@ceremade.dauphine.fr` (Anne-Laure Dalibard).

scalar conservation law :

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) - \varepsilon \Delta u^\varepsilon = 0 \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{1}$$

$$u^\varepsilon(t=0) = u_0\left(x, \frac{x}{\varepsilon}\right). \tag{2}$$

The functions $A_i = A_i(y, v)$ $(y \in Y,\ v \in \mathbb{R})$ are assumed to be Y-periodic, where $Y = \Pi_{i=1}^N (0, T_i)$ is the unit cell, and $u_0 \in L^\infty(\mathbb{R}^N \times Y)$ is also Y-periodic (in fact, a little more regularity is necessary in order to ensure that $u_0\left(x, \frac{x}{\varepsilon}\right)$ is measurable, see for instance section 5 in [1]).

Our goal is to derive the homogenized problem, i.e. to show that there exists a function $u^0 = u^0(t,x,y)$ such that as $\varepsilon \to 0$,

$$u^\varepsilon(t,x) \to u^0(t,x,y)$$

(the precise meaning of the above convergence will be made clear later on) and to find the equations solved by u^0. The homogenized operator can be computed by means of a formal double-scale expansion (see [2]), as we shall see in the second section; our main result is that the y-average of u^0 is the solution of a hyperbolic scalar conservation law, the flux of which can be computed in terms of A and of the solution of a quasilinear elliptic cell problem.

Notice that the viscosity has the same order of magnitude than the size of the heterogeneities, characterized by the small parameter ε; hence, the problem we study in this article is closer to the homogenization of conservation laws and transport equations than to the homogenization of parabolic equations in which the viscosity is of order 1; therefore, the technique we shall use for the proof is inspired from the one developed by W. E and D. Serre in [3] (see also [4], [5], and [6] for an equivalent formulation using Hamilton-Jacobi equations) for the homogenization of a one-dimensional conservation law. From a mathematical point of view, the role of the viscosity here is to simplify the analysis of the cell problem, but it is not fundamental in the convergence proof. Speaking in more physical terms, we will see that viscosity has an effect at a microscopic level only. This is obvious when looking at the homogenized problem : the cell equation, which rules the microscopic behavior of u^0, remains elliptic, while the viscosity vanishes from the macroscopic evolution equation, which is a hyperbolic conservation law.

The proof of our main result relies on the use of two-scale convergence, which was introduced by Allaire in [1], following an idea of Nguetseng (see [7]). The fundamental idea of Allaire and Nguetseng is to try and justify the formal two-scale expansions

$$u^\varepsilon(x) = u^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

widely used in homogenization theory by expressing u^0 as a particular weak limit : precisely, let us recall the basic result of two-scale convergence (see [1]) :

Proposition 1 *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a bounded sequence of $L^2(\Omega)$, where Ω is an open set of $\mathbb{R}^N$. Then as $\varepsilon \to 0$, there exists a subsequence, still denoted by ε, and $u^0 \in L^2(\Omega \times Y)$, such that*

$$\int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(x)\, dx \to \int_{\Omega\times Y} \psi(x,y) u^0(x,y)\, dx\, dy$$

for all $\psi \in \mathcal{C}_{per}(Y, L^2(\Omega))$

Two-scale convergence is thus based on an appropriate choice of oscillating test functions (see also [8] for a variant of this method applied to Hamilton-Jacobi equations, and [9] for an exposition of Tartar's method of oscillating test functions). Unfortunately, we will not be able to use this theorem in the form given by Allaire because of the non-linearity of equation (1); instead, we will need two-scale Young measures, a tool introduced by Weinan E in [4] which handles non-linearities and in which the information contained in two-scale limits is included. We will give more details about two-scale Young measures and their properties in the third section.

Throughout this article, we use the notation

$$\langle v \rangle_Y := \frac{1}{|Y|} \int_Y v(y)\, dy,$$

and we will work in the following functional spaces : if $\mathcal{C}^\infty_{\text{per}}(Y)$ denotes the space of Y-periodic functions in $\mathcal{C}^\infty(\mathbb{R}^N)$, then :

$$\begin{aligned}
H^1_{\text{per}}(Y) &:= \overline{\mathcal{C}^\infty_{\text{per}}(Y)}^{H^1(Y)}, \ ||\cdot||_{H^1_{\text{per}}(Y)} = ||\cdot||_{H^1(Y)}, \\
V &:= \{v \in H^1_{\text{per}}(Y), \langle v\rangle_Y = 0\}, \ ||v||_V = ||\nabla v||_{L^2(Y)} \\
\mathcal{C}^\infty_{\text{per}}(Y\times\mathbb{R}) &:= \{f = f(y,v) \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}); f \text{ is } Y-\text{periodic in } y\}, \\
W^{k,\infty}_{\text{per}}(Y\times\mathbb{R}) &:= \overline{\mathcal{C}^\infty_{\text{per}}(Y\times\mathbb{R})}^{W^{k,\infty}(Y\times\mathbb{R})}, \quad k \in \mathbb{N}, \\
W^{1,\infty}_{\text{per,loc}}(Y\times\mathbb{R}) &:= \{u = u(y,v) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^{N+1}), u \text{ is } Y-\text{periodic in } y\}, \\
K_{\text{per}} &:= \{v(x,y) \in \mathcal{C}^\infty(\mathbb{R}^N\times Y); v \text{ is } Y-\text{periodic in } y \text{ and has compact support in } x\}, \\
J_{\text{per}} &:= \{v(t,x,y) \in \mathcal{C}^\infty([0,+\infty)\times\mathbb{R}^N\times Y); v\ Y-\text{periodic and with compact support in } t,x\}.
\end{aligned}$$

Thanks to the Poincaré-Wirtinger inequality, the norm on V is equivalent to the H^1 norm.

We will often use the following notations:

$$a_i(y,v) := \frac{\partial A_i(y,v)}{\partial v} \ \ (1 \le i \le N), \quad a_{N+1}(y,v) := -\sum_{i=1}^N \frac{\partial A_i(y,v)}{\partial y_i}.$$

The organization of the paper is as follows : in the next subsection, we state our main results, which consist in two theorems : theorem 2 states the existence and uniqueness of solutions of the cell problem, and theorem 3 gives the strong convergence of the sequence u^ε in case of well-prepared initial data. In the next section we derive the homogenized problem thanks to formal double-scale expansions, and we perform the analysis of the cell problem (8). In the third and last section, we give two proofs of theorem 3, the first one using the L^1 contraction principle for equation (1), but requiring very strong regularity assumptions, and the second one using two-scale Young measures.

1.1 Main results

Theorem 2 *Let $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$. Assume that there exist $C_0 > 0$, $m \in [0, \infty)$, $n \in [0, \frac{N+2}{N-2})$ when $N \geq 3$, such that for all $(y, p) \in Y \times \mathbb{R}$*

$$|a_i(y,p)| \leq C_0\,(1 + |p|^m) \quad \forall\, 1 \leq i \leq N, \tag{3}$$

$$|a_{N+1}(y,p)| \leq C_0\,(1 + |p|^n)\,. \tag{4}$$

Assume as well that one of the following conditions holds:

$$m = 0 \tag{5}$$

$$\text{or} \quad 0 \leq n < 1 \tag{6}$$

$$\text{or} \quad n < \frac{N+2}{N} \ \text{ and } \exists p_0 \in \mathbb{R},\ \forall y \in Y\ a_{N+1}(y, p_0) = 0. \tag{7}$$

Then for all $p \in \mathbb{R}$, there exists a unique solution $\tilde{u} \in V$ of the cell problem

$$-\Delta_y \tilde{u} + \operatorname{div}_y A(y, p + \tilde{u}) = 0; \tag{8}$$

For all $p \in \mathbb{R}$, $\tilde{u}(\cdot, p)$ belongs to $W^{2,q}_{per}(Y)$ for all $1 < q < +\infty$ and satisfies the following a priori estimate for all $R > 0$

$$||\tilde{u}(\cdot, p)||_{W^{2,q}(Y)} \leq C \quad \forall p \in \mathbb{R}, |p| \leq R, \tag{9}$$

for some constant C depending only on N, Y, C_0, m, n, q and R.

Theorem 3 *Assume that $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$ satisfies the assumptions of theorem 2, and that $\frac{\partial a_i}{\partial y_j} \in L^1_{loc}(Y \times \mathbb{R})$, $\frac{\partial a_i}{\partial v} \in L^1_{loc}(Y \times \mathbb{R})$ for $1 \leq i \leq N+1$, $1 \leq j \leq N$.*

Let $p \in \mathbb{R}$, and let $\tilde{u}$ be the unique solution in V of the cell problem (8).

Let

$$\bar{A}_i(p) := \frac{1}{|Y|} \int_Y A_i(y, p + \tilde{u}(y, p))\, dy. \tag{10}$$

Assume also that u_0 is "well-prepared", i.e. satisfies

$$u_0(x, y) = v(y, \bar{u}_0(x)) \tag{11}$$

for some $\bar{u}_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$.

Then as ε *goes to 0,*

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0 \quad \text{in } L^2_{loc}([0,\infty)\times\mathbb{R}^N),$$

where $\bar{u} = \bar{u}(t,x) \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty)\times\mathbb{R}^N)$ *is the unique entropy solution of the hyperbolic scalar conservation law*

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \displaystyle\sum_{i=1}^N \dfrac{\partial \bar{A}_i(\bar{u}(t,x))}{\partial x_i} = 0, \\ \bar{u}(t=0,x) = \bar{u}_0(x) \in L^1 \cap L^\infty(\mathbb{R}^N). \end{cases} \tag{12}$$

Remark 4 *Notice that in general, the null function is not a solution of* (1), *unless we make the additional hypothesis* $a_{N+1}(y,0) = 0$ *for all* $y \in Y$. *Therefore, in general there are no global* L^1 *bounds on the solutions of* (1), *even if* $u_0\left(x, \frac{x}{\varepsilon}\right) \in L^1(\mathbb{R}^N)$. *Moreover, slightly stronger assumptions on the flux* A *are required in general in order to ensure the existence of solutions of* (1), *e.g.* $A \in W^{2,\infty}_{per}(Y \times \mathbb{R})$. *The hypothesis* $\frac{\partial a_i}{\partial y_j}, \frac{\partial a_i}{\partial v} \in L^1_{loc}(Y \times \mathbb{R})$ *is necessary so that the* L^1 *contraction principle holds.*

Remark 5 *Assumption* (11) *means that the initial data is already adapted to the microstructure; if it is not, i.e. if it cannot be written in the form*

$$u_0(x,y) = v\left(y, \bar{u}_0(x)\right),$$

then it is expected that there will be an initial layer of order ε *during which the solution will adjust itself to the microstructure; this problem is not addressed here, and will be delt with in a future article.*

2 Formal computation of the homogenized problem

In order to compute the effective equations which rule the system in the limit $\varepsilon \to 0$, we use double scale asymptotic expansions (see [2] for a general presentation of this technique): assume that u^ε satisfies the following Ansatz :

$$u^\varepsilon(t,x) = u^0\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t,x,\frac{x}{\varepsilon}\right) + \cdots.$$

Inserting this expansion in equation (1) and identifying the powers of ε, we derive the following equations on u^0, u^1 :

$$\text{Order } \varepsilon^{-1} \; : \quad -\Delta_y u^0(t,x,y) + \text{div}_y A(y, u^0(t,x,y)) = 0, \tag{13}$$

$$\text{Order } \varepsilon^{0} \; : \quad \frac{\partial u^0}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[A_i(y,u^0)\right] - 2\Delta_{xy}u^0 - \Delta_y u^1 + \sum_{i=1}^{N} \frac{\partial}{\partial y_i}\left[u^1 a_i(y,u^0)\right] = 0. \tag{14}$$

(13) leads us to write u^0 in the form

$$u^0(t,x,y) = \bar{u}(t,x) + \tilde{u}(y, \bar{u}(t,x)),$$

where $\bar{u}(t,x) := \langle u^0(t,x,\cdot)\rangle_Y$ and $\tilde{u} = \tilde{u}(y,p), \;\; y \in Y, p \in \mathbb{R}$ satisfies the so-called cell equation

$$-\Delta_y \tilde{u} + \text{div} A(y, p + \tilde{u}(y,p)) = 0$$

together with the condition $\langle \tilde{u} \rangle_Y = 0$ for all p. Then, averaging (14) with respect to y yields the evolution equation on $\bar{u}$:

$$\frac{\partial \bar{u}}{\partial t} + \sum_{i=1}^{N} \frac{\partial \bar{A}_i(\bar{u})}{\partial x_i} = 0,$$

where the homogenized flux $\bar{A}_i$ can be computed thanks to the formula

$$\bar{A}_i(p) := \langle A(\cdot, p + \tilde{u}(\cdot,p))\rangle_Y \,.$$

The ε^0 term also allows us to derive the equation on u^1 :

$$-\Delta_y u^1 + \sum_{i=1}^{N} \frac{\partial}{\partial y_i}\left[u^1 a_i(y,u^0)\right] = 2\Delta_{xy}u^0 - \left\{ \frac{\partial u^0}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[A_i(y,u^0)\right] \right\}.$$

Unfortunately, these calculations are entirely formal, and must be justified rigorously. In the following subsections, we will show that the homogenized equations computed above have solutions, and in the third section, we shall prove the convergence of u^ε to the solution of the homogenized problem.

2.1 Cell problem

This subsection is devoted to the proof of theorem 2. In fact, more general results can be proved, which we state in the following lemmas.

Lemma 6 *Assume $A \in W^{1,\infty}_{per,\,loc}(Y \times \mathbb{R})$ satisfies (3),(4) with $m \geq 0$ arbitrary, $n \in [0, \frac{N+2}{N-2})$ when $N > 2$ (if $N \leq 2$, there is no restriction on n).*

(1) Regularity: If $\tilde{u} \in H^1_{per}(Y)$ is a solution of (8) for some $p \in \mathbb{R}$, then $\tilde{u} \in W^{2,q}(Y)$ for all $1 < q < +\infty$, and the following estimate holds: for all $R > 0$, there exists a constant $C > 0$ depending only on R, q, N, Y, m, n and C_0, and a constant M depending only on q, m, n and N such that

$$||\tilde{u}(\cdot, p)||_{W^{2,q}(Y)} \leq C(1 + ||\tilde{u}||_{H^1(Y)})^M \quad \forall p \in [-R, R]. \tag{15}$$

(2) Uniqueness and monotony: for all $p \in \mathbb{R}$, there exists at most one solution $\tilde{u}(y, p) \in V$ of (8). Moreover, if $\tilde{u}(y, p)$ and $\tilde{u}(y, p')$ are two solutions of (8) with $p \geq p'$, then setting $v(y, p) := p + \tilde{u}(y, p)$ we have

$$v(y, p) \geq v(y, p') \quad a.e. \text{ on } Y.$$

(3) p-derivative : assume that there exists a solution of (8) for all $p \in \mathbb{R}$ and that

$$K_R := \sup_{|p| \leq R} ||\tilde{u}(\cdot, p)||_{H^1(Y)} < +\infty \quad \forall R > 0.$$

Then for all $p \in \mathbb{R}$, $\frac{\partial \tilde{u}}{\partial p}(\cdot, p) \in H^1_{per}(Y)$ and for all $R > 0$ there exists $C = C(R, N, Y, C_0, m, n, K_R)$ such that

$$\left|\left|\frac{\partial \tilde{u}}{\partial p}\right|\right|_{L^\infty((-R,R), H^1_{per}(Y))} \leq C. \tag{16}$$

Moreover, $\frac{\partial v}{\partial p} \in H^1_{per}(Y)$ is the unique solution of

$$-\Delta_y \frac{\partial v}{\partial p} + \operatorname{div}_y \left(a(y, v(y, p)) \frac{\partial v}{\partial p}\right) = 0 \tag{17}$$

under the constraint $\left\langle \frac{\partial v}{\partial p} \right\rangle_Y = 1$.

The Krein-Rutman theorem ensures that

$$\frac{\partial v}{\partial p}(y, p) > 0 \qquad \text{for a.e. } (y, p) \in Y \times \mathbb{R}. \tag{18}$$

If additionnally $a_i \in L^\infty(Y \times \mathbb{R})$ for $1 \leq i \leq N$ (i.e. $m = 0$), then there exists $\alpha > 0$ depending only on N, Y, and $\max_{1 \leq i \leq N} ||a_i||_\infty$ such that

$$\frac{\partial v(y, p)}{\partial p} \geq \alpha > 0 \quad \forall y \in Y \ \forall p \in \mathbb{R}. \tag{19}$$

Hence

$$\inf_Y v(y, p) \to +\infty \quad \text{as } p \to +\infty, \tag{20}$$

$$\sup_Y v(y, p) \to -\infty \quad \text{as } p \to -\infty. \tag{21}$$

We now state the existence result:

Lemma 7 *Assume* $A \in W^{1,\infty}_{per,\, loc}(Y \times \mathbb{R})$ *satisfies* (3),(4) *with* m *and* n *satisfying one of the three conditions* (5), (6) *or* (7). *Then there exists a (unique) solution of* (8) *for all* $p \in \mathbb{R}$, *and it satisfies the following a priori estimate*

$$||\tilde{u}(p)||_{H^1} \leq C_R \quad \forall p \in [-R, R], \tag{22}$$

where C_R *depends on*

(1) N, Y, *and* C_0 *when* (5) *is satisfied;*
(2) N, Y, C_0, R *and* n *when* (6) *is satisfied;*
(3) N, Y, C_0, R, n *and* p_0 *when* (7) *is satisfied.*

Remark 8 *Hypothesis* (7) *can be slightly relaxed : in fact, we only need that for all* $\lambda \in [0, 1]$, *there exists* $p_\lambda \in \mathbb{R}$ *and* $u_\lambda \in V$ *such that*

$$-\Delta_y u_\lambda + \lambda \mathrm{div}_y A(y, p_\lambda + u_\lambda) = 0$$

and $\sup_{\lambda \in [0,1]}(|p_\lambda| + ||u_\lambda||_{L^1(Y)}) < +\infty$.

In that case, the constant C_R *in the a priori estimate* (22) *depends on* N, Y, C_0, R, n *and* $\sup_{\lambda \in [0,1]}(|p_\lambda| + ||u_\lambda||_{L^1(Y)})$.

If $a_{N+1}(y, p_0) \equiv 0$, *we can take* $p_\lambda = p_0$ *for all* $\lambda \in [0, 1]$, *and* $u_\lambda \equiv 0$.

We will need the following lemma, of which we skip the proof :

Lemma 9 *Let* $b \in L^\infty(Y)^N$, $\alpha > 0$, $f \in L^2(Y)$. *Let* $m \in H^1_{per}(Y)$ *be a solution of*

$$-\Delta_y m + \mathrm{div}_y(bm) = f$$

such that $|\int_Y m| \leq \alpha$.

There exists a positive constant C, *depending only on* N, Y, $||b||_{L^\infty(Y)^N}$, $||f||_{L^2(Y)}$ *and* α, *s.t.*

$$||m||_{H^1(Y)} \leq C.$$

Proof of Lemma 6

- *First step : A priori* estimates :

Multiplying equation (8) by $|\tilde{u}|^{q-1}\tilde{u}$, for some $q \geq 1$, we see that if $\tilde{u} \in V \cap L^{n+q}$ is a solution of (8), then $\tilde{u}$ satisfies

$$q \int_Y |\nabla \tilde{u}|^2 |\tilde{u}|^{q-1}\, dy = q \int_Y |\tilde{u}|^{q-1} A(y, p + \tilde{u}) \cdot \nabla \tilde{u}\, dy;$$

set

$$B_i(y,w) = \int_0^w |r|^{q-1} A_i(y, p+r)\, dr \qquad \text{for } 1 \le i \le N.$$

Then using hypothesis (4)

$$\begin{aligned}
q\int_Y |\nabla \tilde{u}|^2 |\tilde{u}|^{q-1}\, dy &= \underbrace{q\sum_{i=1}^N \int_Y \frac{\partial}{\partial y_i}\left[B_i(y,\tilde{u}(y))\right]\, dy}_{=0} - q\int_Y \sum_{i=1}^N \frac{\partial B_i}{\partial y_i}(y,\tilde{u}(y))\, dy \\
&= q\int_Y \int_0^{\tilde{u}(y)} |r|^{q-1} a_{N+1}(y,p+r)\, dr \\
&\le qC_0 \int_Y \int_0^{\tilde{u}(y)} |r|^{q-1}\left(1 + (|p|+|r|)^n\right)\, dr\, dy \\
\left\|\nabla\left(\tilde{u}^{\frac{q+1}{2}}\right)\right\|_{L^2(Y)} &\le C\left((1+|p|)^{\frac{n}{2}} ||\tilde{u}||_{L^q}^{\frac{q}{2}} + ||\tilde{u}||_{L^{n+q}}^{\frac{n+q}{2}}\right), \qquad (23)
\end{aligned}$$

for all $q \ge 1$ and for some constant C depending only on N, n, Y, C_0 and q.

- *Second step :* $\tilde{u} \in \cap_{1\le r<+\infty} L^r(Y)$:

Let $R > 0$ arbitrary, and let $p \in [-R, R]$, $n_0 = \max(1, n)$. According to the a priori estimate (23), there exists a constant C_R depending only on R, N, n, Y, C_0 and q such that if $\tilde{u} \in V \cap L^{q+n_0}(Y)$ is a solution of (8)

$$||\tilde{u}^{\frac{q+1}{2}}||_{H^1} \le C_R \left(1 + ||\tilde{u}||_{L^{q+n_0}}\right)^{\frac{q+n_0}{2}}$$

H^1 is imbedded in $L^{\frac{2N}{N-2}}(Y)$ for $N > 2$, and in $L^r(Y)$ for $N \le 2$, $1 \le r < +\infty$ arbitrary. Hence if $\tilde{u} \in V$ is a solution of (8)

$$\begin{gathered}
\tilde{u} \in \cap_{1\le r<+\infty} L^r(Y) \quad \text{if } N \le 2, \\
\tilde{u} \in L^{q+n_0}(Y) \Rightarrow \tilde{u} \in L^{\frac{(q+1)N}{N-2}}(Y) \quad \forall q \in [1, +\infty) \text{ if } N > 2.
\end{gathered}$$

When $N > 2$, define the sequence $(q_k)_{k\ge1}$ by

$$q_1 = 1, \quad q_{k+1} + n_0 = (q_k + 1)\frac{N}{N-2}, \quad k \ge 1.$$

Then it is easily checked that since $n < \frac{N+2}{N-2}$, $q_k \ge 1$ for all $k \ge 1$ and

$$\begin{aligned}
&u \in L^{q_k+n}(Y) \quad \forall k \ge 1, \\
&q_k \to +\infty \quad \text{as } k \to \infty.
\end{aligned}$$

Moreover,

$$||\tilde{u}||_{L^{1+n_0}} \le C||\tilde{u}||_{L^{\frac{2N}{N-2}}} \le C||\tilde{u}||_{H^1}$$

where the constant C depends only on N, Y, and n.

In all cases, $\tilde{u} \in \cap_{1\leq r<+\infty} L^r(Y)$. And for all $r \geq 2$, there exists a constant C_R depending only on R, r, N, n, C_0 and Y, and a constant M depending only on r, n and N such that for all $p \in [-R, R]$, for all solutions $\tilde{u} \in V$ of (8)

$$||\tilde{u}||_{L^r} \leq C_R \left(1 + ||\tilde{u}||_{H^1}\right)^M . \tag{24}$$

- *Third step :* $W^{2,r}$ estimates :

Let $R > 0$, and let $p \in \mathbb{R}$, $|p| \leq R$; let $\tilde{u}$ be a solution of (8) for the parameter p.

Since $\tilde{u} \in H^1_{\text{per}}(Y)$, the chain rule allows us to write

$$-\Delta_y \tilde{u} = a_{N+1}(y, p + \tilde{u}(y)) - a(y, p + \tilde{u}(y)) \cdot \nabla_y \tilde{u}. \tag{25}$$

In the above equation, $a_{N+1}(y, p + \tilde{u}(y))$, $a(y, p + \tilde{u}(y))$ belong to $L^r(Y)$ for all $r \in [1, +\infty)$, and $\nabla_y \tilde{u} \in L^2(Y)$.

Hence the right-hand side belongs to $L^q(Y)$ for all $1 < q < 2$, with locally uniform bounds in p. Using interior regularity results for elliptic equations (see [10],[11]) combined with the periodicity, it can be proved that $\tilde{u} \in W^{2,q}(Y)$ for all $q < 2$ and

$$||\tilde{u}||_{W^{2,q}(Y)} \leq C(1 + ||\tilde{u}||_{H^1})^M, \tag{26}$$

for a constant C depending only on C_0, m, n, N, Y, q, and R and a constant M depending only on m, n, N and q.

Next, Sobolev imbeddings entail that $\nabla \tilde{u} \in L^\infty_{\text{loc}}(\mathbb{R}, L^q(Y))$ for all $q > 1$ such that $\frac{1}{q} > \frac{1}{2} - \frac{1}{N}$ and we can repeat the same argument as above replacing 2 by $\frac{2N}{N-2}$ (if $N > 2$).

More precisely, let us define the sequence q_k by

$$\frac{1}{q_k} = \frac{1}{2} - \frac{k}{N} \quad \text{if } k < \frac{N}{2};$$

then it is easily checked using the above method that

$$\tilde{u} \in W^{1,q}(Y) \ \forall q \in (1, q_k) \Rightarrow \tilde{u} \in W^{1,q}(Y) \ \forall q \in (1, q_{k+1}),$$

as long as $k + 1 < \frac{N}{2}$, and with bounds of the type (26).

By induction, $\tilde{u} \in W^{1,q}(Y)$ for all $1 < q < q_{k_0}$, where k_0 is the integer defined by

$$k_0 < \frac{N}{2} \leq k_0 + 1.$$

Then $q_{k_0} \geq N$; consequently, $\tilde{u} \in W^{2,q}(Y)$ for all $q < N$, and thus $\tilde{u} \in W^{1,r}(Y)$ for all $r \geq 1$. Plugging this result once more into (25) yields $\tilde{u} \in W^{2,r}$ for all $r \geq 1$, with bounds of the type (26). Hence (15) is proved.

- *Fourth step :* Uniqueness and monotony of solutions of (8) :

If $\tilde{u}_1$ and $\tilde{u}_2$ are two solutions of (8) for parameters p_1, p_2, then $w_{p_1,p_2} := (p_1+\tilde{u}_1) - (p_2+\tilde{u}_2) \in V$ satisfies an elliptic equation

$$-\Delta_y w_{p_1,p_2} + \mathrm{div}_y(b_{p_1,p_2} w_{p_1,p_2}) = 0,$$

where

$$b_{p_1,p_2}(y) = \int_0^1 a(y,(1-\tau)v(y,p_1)+\tau v(y,p_2))\, d\tau.$$

Thanks to the regularity result we have just shown, $b_{p_1,p_2} \in L^\infty(Y)^N$ for all $p_1, p_2 \in \mathbb{R}$. And for all $R>0$, there exists a constant C depending on N, Y, C_0, m, n, R, $||\tilde{u}(p_1)||_{H^1}$, $||\tilde{u}(p_2)||_{H^1}$, such that

$$||b_{p_1,p_2}||_{L^\infty(Y)^N} \le C \quad \forall p_1,p_2 \in [-R,R].$$

The uniqueness and the monotony follow from the following lemma

Lemma 10 *Let $b \in L^\infty(Y)^N$, and let $v \in H^1_{per}(Y)$ be a solution of the linear elliptic equation*

$$-\Delta_y v + \mathrm{div}_y(bv) = 0. \tag{27}$$

There exists a positive probability measure $m \in M^1_{per}(Y) = \mathcal{C}_{per}(Y)'$ and a constant $c \in \mathbb{R}$ such that $v = cm$. In particular, if $\langle v\rangle_Y = 0$, then $v = 0$.

We postpone the proof of the lemma.

Hence, since $\langle w_{p_1,p_2}\rangle_Y = (p_1-p_2)$, we deduce that $w_{p_1,p_2} = (p_1-p_2)m_{p_1,p_2}$, with m_{p_1,p_2} a positive measure on Y. If $p_1 = p_2$, then $w_{p_1,p_2} = 0$, and the uniqueness is proved. If $p_1 > p_2$, then

$$v(y,p_1) > v(y,p_2) \quad \forall y \in Y.$$

As a consequence, we deduce

$$||v(y,p_1) - v(y,p_2)||_{L^1(Y)} = \int_Y (v(y,p_1)-v(y,p_2))\ dy = |Y|(p_1-p_2)$$

- *Fifth step :* p-derivative:

Now, $m_{p_1,p_2}(y) = \frac{v(y,p_1)-v(y,p_2)}{p_1-p_2}$ is a positive measure on Y for $p_1 \neq p_2$, $p_1,p_2 \in \mathbb{R}$, and m_{p_1,p_2} satisfies

$$-\Delta_y m_{p_1,p_2} + \mathrm{div}_y(b_{p_1,p_2} m_{p_1,p_2}) = 0, \quad \langle m_{p_1,p_2}\rangle = 1. \tag{28}$$

Assume that

$$K_R := \sup_{|p|\le R} ||\tilde{u}(p)||_{H^1(Y)} < +\infty \quad \forall R>0.$$

Then for all $R > 0$ there exists a constant $C_R > 0$ depending only on N, Y, n, m, C_0, R and K_R such that

$$||b_{p_1,p_2}||_{L^\infty(Y)^N} \leq C_R \quad \forall p_1, p_2 \in [-R, R].$$

Hence, using lemma 9, there exists a positive constant $C = C(R, C_0, m, n, N, Y, K_R)$ such that

$$||m_{p_1,p_2}||_{H^1(Y)} \leq C \quad \forall (p_1, p_2) \in \mathbb{R}^2,\ p_1 \neq p_2, |p_1|, |p_2| \leq R.$$

Let $p_n \to p_0$, $p_0 \in [-R, R]$. Extracting a subsequence, $m_{p_0,p_n}(\cdot)$ converges weakly in $H^1_{\text{per}}(Y)$, strongly in $L^2(Y)$ to $\frac{\partial v}{\partial p}(y, p_0)$ and b_{p_n,p_0} converges to $a(y, v(y, p_0))$. Passing to the limit in equation (28) leads to equation (17). *A priori* estimates are obtained using lemma 9, and eventually, lemma 10 entails that $\frac{1}{|Y|}\frac{\partial v}{\partial p}(y, p_0)$ is a positive probability measure on Y for all p_0.

- *Sixth step :* Proof of (20), (21) :

When $m = 0$, $\frac{\partial v}{\partial p}$ satisfies (17) with

$$||a(y, v(y, p))||_{L^\infty(Y)^N} \leq 2C_0 \quad \forall p \in \mathbb{R}.$$

According to the Harnack inequality (see for instance [10]) combined with the periodicity, there exists a constant C depending only on C_0, N, and Y such that

$$\sup_Y w \leq C \inf_Y w.$$

Since $\int_Y w = |Y|$, $\sup_Y w \geq 1$. Hence there exists a positive constant α, depending only on C_0, N, and Y, such that

$$\frac{\partial v(y, p)}{\partial p} \geq \alpha$$

and (20), (21) are proved.

Proof of Lemma 7

Let us define the operator $T : u \in V \mapsto v \in V$ where $v = T(u) \in V$ is the unique solution of the elliptic equation

$$-\Delta_y v = -\text{div}_y A(y, p + u(y)).$$

Fixed points of T are solutions of (8), and T is a continuous compact operator.

We want to apply Schaefer's fixed point theorem, and thus prove that

$$\{u \in V; \exists \lambda \in [0, 1],\ u = \lambda T(u)\}$$

is bounded. In the sequel, we take $u \in V$, $\lambda \in [0,1]$ such that $u = \lambda T(u)$, and we try and derive a bound on u.

We begin with the case $m = 0$. In that case, u satisfies

$$-\Delta_y u + \operatorname{div}(bu) = \lambda a_{N+1}(y,0),$$

where

$$b(y) = \lambda \int_0^1 a(y, t(p+u))\, dt.$$

Hence $b \in L^\infty(Y)^N$ and

$$\begin{aligned} ||b||_{L^\infty(Y)^N} &\le ||a||_{L^\infty(Y\times\mathbb{R})^N} \le 2C_0, \\ ||\lambda a_{N+1}(y,0)||_{L^2(Y)} &\le C_0 |Y|^{\frac{1}{2}}. \end{aligned}$$

Thus according to lemma 9, there exists a constant C depending only on N, Y and C_0, such that

$$||u||_{H^1(Y)} \le C$$

and the estimate is proved.

When either (6) or (7) are satisfied, for all $u \in V$ such that $u = \lambda T(u)$, the a priori estimate (23) with $q = 1$ and changing A into λA yields

$$||u||_{H^1(Y)} \le C \left((1+|p|)^{\frac{n}{2}} ||u||_{L^1}^{\frac{1}{2}} + ||u||_{L^{n+1}}^{\frac{n+1}{2}} \right) \tag{29}$$

for some constant C depending only on N, n, C_0 and Y.

If $n < 1$, then it is easily seen that this inequality leads to an H^1 a priori estimate, and thus to the existence of solutions of (8). Hence, we now focus on the case $n \ge 1$.

Since $n + 1 < \frac{2N+2}{N} \le \frac{2N}{N-2}$ (if $N > 2$), we can interpolate L^{n+1} between L^1 and $L^{\frac{2N}{N-2}}$: let $\theta \in (0,1]$ such that

$$\frac{1}{n+1} = \frac{\theta}{1} + \frac{1-\theta}{q_0}$$

where $q_0 := \frac{2N}{N-2}$. Then

$$\begin{aligned} ||u||_{L^{n+1}}^{\frac{n+1}{2}} &\le ||u||_{L^1}^{\frac{(n+1)\theta}{2}} ||u||_{L^{q_0}}^{\frac{(n+1)(1-\theta)}{2}} \\ &\le ||u||_{L^1}^{\frac{(n+1)\theta}{2}} ||u||_{H^1}^{\frac{(n+1)(1-\theta)}{2}}. \end{aligned}$$

It is easily checked that $n < \frac{N+2}{N}$ if and only if $\frac{(n+1)(1-\theta)}{2} < 1$. The whole problem thus reduces to find L^1 estimates for the solutions of (8). This is quite easy if hypothesis (7) is satisfied. Indeed, in that case, $\tilde{u}(y,p_0) \equiv 0$ is a special solution of (8) for $p = p_0$ and for the flux λA; hence, according to lemma 6,

$$||u||_{L^1} = ||u - \tilde{u}(p_0)||_{L^1} \le |p-p_0||Y| + ||(p+u) - (p_0 + \tilde{u}(p_0))||_{L^1} \le 2|p-p_0||Y|.$$

Plugging these estimates into (29) yields

$$||u||_{H^1} \leq C_R \left(1 + ||u||_{H^1}^{\frac{(n+1)(1-\theta)}{2}}\right) \tag{30}$$

for all p such that $|p| \leq R$, where the constant C_R depends only on N, Y, n, C_0, p_0 and R. Since $\frac{(n+1)(1-\theta)}{2} < 1$, $||u||_{H^1}$ is bounded by a constant depending on the same parameters as C_R. Hence the a priori estimate is proved and solutions of (8) exist for all $p \in \mathbb{R}$.

Proof of Lemma 10

The constant function equal to 1 on Y, denoted by $\bar{1}$, is a solution of the dual problem

$$-\Delta_y \bar{1} - b(y) \cdot \nabla_y \bar{1} = 0. \tag{31}$$

We want to prove, using the strong form of the Krein-Rutman theorem, that there exists a constant $c \in \mathbb{R}$ such that $w = cm$, where $m > 0$ is a solution of (27). Indeed, in that case $c = 0$ necessarily since $\langle w \rangle_Y = 0$ and thus $w = 0$.

Let us introduce the operator $F : u \in L^2(Y) \mapsto v \in H$ where $v = F(u)$ is the unique solution of the equation

$$-\Delta v - b \cdot \nabla v + \alpha v = \alpha u,$$

and α is a positive constant chosen so that the bilinear form associated to F is coercive (e.g. $\alpha = \frac{||b||_\infty^2}{2} + \frac{1}{2}$). With that choice of α F is a strictly positive operator.

Next, using once again interior regularity results for linear elliptic equations combined with the periodicity, we show that F maps $L^q(Y)$ into $W^{2,q}_{\mathrm{per}}(Y)$ for all $q \geq 2$. Hence, the restriction of F to $\mathcal{C}_{\mathrm{per}}(\overline{Y})$, still denoted by F, is a compact operator from $\mathcal{C}_{\mathrm{per}}(\overline{Y})$ into itself.

The last step consists in using the maximum principle: if $u \in \mathcal{C}_{\mathrm{per}}(\overline{Y})$, $u \geq 0$, $u \neq 0$ and $v = F(u)$, then $v(y) > 0$ for all $y \in \overline{Y}$ (see for instance [12]; the maximum principle is in general proved for classical solutions of elliptic equations with regular coefficients. However the proofs remain unchanged for weak solutions and $b \in L^\infty$ provided the following property holds true for any $\gamma > 0$:

$$u \in L^2(Y), u \geq \gamma \quad \text{a.e.} \Rightarrow v = F(u) \geq \gamma.$$

This property can be proved by approximating b in L^q for $1 < q < \infty$ by a sequence $b_n \in \mathcal{C}^\mu(\overline{Y})$ for some $\mu \in (0,1)$.)

Hence, $F : \mathcal{C}_{\mathrm{per}}(\overline{Y}) \to \mathcal{C}_{\mathrm{per}}(\overline{Y})$ is a strongly positive operator.

We conclude by using the strong form of the Krein-Rutman theorem (see [13], [14]): since $F(\bar{1}) = \bar{1}$, the spectral radius of F is equal to 1 and 1 is a simple eigenvalue of F^*, the adjoint of F, with a positive eigenvector. Let $m \in M^1_{\mathrm{per}}(Y) = \mathcal{C}_{\mathrm{per}}(Y)'$ be the unique positive invariant measure such that $\langle m\rangle_Y = 1$ and $F^*(m) = m$. Since $v \in H^1_{\text{per}}(Y) \subset M^1_{\text{per}}(Y)$ solves (27), $F^*(v) = v$; thus, there exists $c \in \mathbb{R}$ such that $v = cm$. If $\langle v\rangle_Y = 0$, then $c = 0$ and $v = 0$, which completes the proof of the lemma.

Remark 11 *This lemma can be generalized without any difficulty to the case $b \in L^q$ for some $q > N$ using the inequality*

$$\left|\int_Y vb \cdot \nabla v\right| \le C||b||_{L^q}||v||_{L^2}^{1-\frac{N}{q}}||\nabla v||_{L^2}^{1+\frac{N}{q}}$$

where C is a constant depending only on N and Y.

Remark 12 *Let us point out that the techniques we have used in order to find a priori bounds on the solutions of the cell problem rely strongly on the ellipticity of equation* (8)*. In particular, when the viscosity is equal to 0 in equation* (1)*, the cell problem becomes*

$$\text{div}_y A(y, p + \tilde{u}(y)) = 0,$$

and we have no clue how to derive a priori bounds on the solutions of the above equation in general. The few cases in which we are able to prove such bounds suggest strongly that the flux A should be nonlinear. However, it is an open problem how to treat such an equation in general, and which hypotheses should be expected on the flux. We will come back on these questions in a future paper.

Before going any further in the multi-scale analysis of problem (1), let us mention a few examples in which hypotheses (5), (6), and (7) seem "natural".

Take $A(y, v) = b(y)f(v)$, where $b \in W^{1,\infty}(Y)^N$ has values in $\mathbb{R}^N$, $f \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ is scalar.

If $\text{div}_y b \equiv 0$ on Y, then constants are solutions of equation (8). Lemma 6 asserts that there are no other solutions as long as f has polynomial growth. Notice that in that case, hypothesis (7) is satisfied.

Let us study now the less trivial case $b(y) = \nabla_y \phi(y)$, where $\phi \in \mathcal{C}^1_{\text{per}}(Y)$. Assume that f does not vanish on $\mathbb{R}$ (otherwise we are in case (7)); without loss of generality, we can assume that

$$f(v) > 0 \quad \forall v \in R.$$

We can thus define

$$H(v) := \int_0^v \frac{1}{f(w)}\, dw \quad \forall w \in \mathbb{R}.$$

It is obvious that any solution of

$$-\nabla_y u = -\nabla_y \phi(y) f(p + u) \tag{32}$$

is a solution of (8); hence, we search for particular solutions of (8) which satisfy (32).

(32) is equivalent to

$$\nabla_y H(p + u) = \nabla \phi,$$

and thus to

$$H(p+u) = \phi + \text{cst}.$$

Thus we deduce that solutions of (32) exist if and only if

$$H(+\infty) - H(-\infty) = \int_{\mathbb{R}} \frac{1}{f} > \text{osc}\phi. \tag{33}$$

In particular, this is always satisfied when $|f(v)| \leq C_0(1+|v|)^n$ for some $n<1$ (i.e. when (6) holds) since in that case

$$H(+\infty) - H(-\infty) = \int_{\mathbb{R}} \frac{1}{f} = +\infty.$$

Assume that (33) is satisfied; notice that $H \in \mathcal{C}^1(\mathbb{R})$ and $H' = \frac{1}{f}$ does not vanish on $\mathbb{R}$. Hence H is a $\mathcal{C}^1$ diffeomorphism from $\mathbb{R}$ to $(H(-\infty), H(+\infty)) =: (\alpha, \beta)$. We denote by $H^{-1} : (\alpha, \beta) \to \mathbb{R}$ its reciprocal application. Let

$$c_+ := \beta - \max \phi,$$
$$c_- := \alpha - \min \phi.$$

Then for all $c \in (c_-, c_+)$ we can define

$$v_c := H^{-1}(\phi + c), \quad u_c := v_c - \langle v_c \rangle \tag{34}$$

and u_c is a solution of (8) for all $c \in (c_-, c_+)$. Hence, when (33) is satisfied, we have found special solutions of (8). If $c_\pm = \pm\infty$, then we have found solutions for all values of the parameter p in (8). If $|f(v)| \leq C_0(1+|v|)^n$ with $n < \frac{N+2}{N}$, then we deduce that there exist solutions of (8) for all values of p as well thanks to lemma 7 and the remark following the lemma (changing A into λA is equivalent to changing ϕ into $\lambda\phi$).

Reciprocally, let us prove that (33) is a necessary condition for solutions of (8) to exist at all when $n < \frac{N+2}{N-2}$. Let $u_0 \in V$ be a solution of (8) for the parameter $p_0 \in \mathbb{R}$, and let $v_0 := p_0 + u_0$. According to lemma (6), $v_0 \in L^\infty(Y)$. Hence we can change the function f for values of v larger than $||v_0||_{L^\infty}$ so that the function $\tilde{f}$ thus obtained satisfies (33) and

$$\tilde{f}(v_0(y)) = f(v_0(y)) \quad \forall y \in Y.$$

We can even choose $\tilde{f}$ so that

$$\int_0^\infty \frac{1}{\tilde{f}} = \int_{-\infty}^0 \frac{1}{\tilde{f}} = +\infty.$$

In that case, we have proved that there exist solutions of (8) for the flux $\nabla\phi(y)\tilde{f}(v)$ for all values of the parameter p. Let u_{c_0} be the solution for the parameter p_0,

$v_{c_0} := u_{c_0} + p_0$. Then

$$-\Delta_y v_{c_0} + \mathrm{div}_y \left(\nabla\phi(y)\tilde{f}(v_{c_0}(y))\right) = 0,$$
$$-\Delta_y v_0 + \mathrm{div}_y \left(\nabla\phi(y)\tilde{f}(v_0(y))\right) = -\Delta_y v_0 + \mathrm{div}_y \left(\nabla\phi(y) f(v_0(y))\right) = 0,$$

and by uniqueness of the solutions of (8) for the flux $\nabla\phi(y)\tilde{f}(v)$, $v_0 = v_{c_0}$. Consequently,

$$v_0 = \tilde{H}^{-1}(\phi + c_0) = H^{-1}(\phi + c_0)$$

and (33) is satisfied. Moreover, we have proved that all solutions of (8) can be written in the form (34).

Now, let us explain why condition (7) is optimal, to a certain extent. Take $f(v) = (1+|v|^2)^{\frac{n}{2}}$ for some $n > \frac{N+2}{N}$. Then $\alpha = -\beta \in \mathbb{R}$. Assume that (33) is satisfied. In order to simplify our analysis, we assume as well that ϕ attains its minimum in a unique point y_0 in the interior of Y.

We define

$$v_- := H^{-1}(\phi + c_-);$$

$v_-(y)$ is finite for all $y \neq y_0$. Moreover, if $u \in V$ is a solution of (8) for the parameter p, then u can be written in the form (34). Thus there exists $c \in (c_-, c_+)$ such that $u + p = v_c$ and necessarily

$$p + u > v_-.$$

Hence, if we can prove that $v_- \in L^1(Y)$, we will be able to derive a lower bound on the admissible values of p so that there exists a solution of (8) for the parameter p. In other words, there will be no solution for $p < \langle v_- \rangle$.

Let us prove that $v_- \in L^1(Y)$: there exists a constant $c \geq 1$ such that for y in a neighbourhood V_0 of y_0

$$\frac{1}{c}|y - y_0|^2 \leq \phi(y) - \phi(y_0) \leq c|y - y_0|^2.$$

Hence

$$\frac{1}{c}|y - y_0|^2 \leq H(v_-) - \alpha \leq c|y - y_0|^2.$$

On the other hand, there exists a constant C depending only on n such that for all $A \geq 1$,

$$\frac{1}{C}\frac{1}{A^{n-1}} \leq \int_A^\infty \frac{1}{f(v)}\, dv \leq C\frac{1}{A^{n-1}}.$$

Choose V_0 such that $v_-(y) \leq -1$ in V_0. In V_0,

$$\frac{1}{C}\frac{1}{|v_-|^{n-1}} \leq H(v_-) - \alpha = \int_{-\infty}^{v_-} \frac{1}{f(v)}\, dv \leq C\frac{1}{|v_-|^{n-1}}.$$

Thus, there exists a constant C such that for all $y \in Y_0$

$$|v_-| \leq \frac{C}{|y - y_0|^{\frac{2}{n-1}}}.$$

If $n > \frac{N+2}{N}$, then $\frac{2}{n-1} < N$ and the singularity in y_0 is integrable: $v_- \in L^1(Y)$.

Let us gather our results in the following

Lemma 13 *Let $A(y,v) = \nabla\phi(y) f(v)$, with $f(v) > 0$ for all $v \in \mathbb{R}$. Assume that*

$$f(v) \leq C_0(1+|v|)^n \quad \text{with } n < \frac{N+2}{N-2}.$$

Then

(1) *There exist solutions of* (8) *for some values (but possibly not all) of the parameter p if and only if*
$$\int_{\mathbb{R}} \frac{1}{f} > \mathrm{osc}\phi.$$
(2) *If the above inequality is satisfied and $f(v) \leq C_0(1+|v|)^n$ with $n < \frac{N+2}{N}$ then there exist solutions of* (8) *for all values of the parameter p.*
(3) *If*
$$\int_0^\infty \frac{1}{f} = \int_{-\infty}^0 \frac{1}{f} = +\infty$$
then there exist solutions of (8) *for all values of $p \in \mathbb{R}$.*
(4) *If $|f(v)| = (1+|v|^2)^{\frac{n}{2}}$ with $n > \frac{N+2}{N}$, then there exists $\phi \in \mathcal{C}^1_{per}(Y)$ and $p_-, p_+ \in \mathbb{R}$ such that there are no solutions of* (8) *for $p < p_-$ or $p > p_+$.*

The second point in the above lemma is the analogue of condition (7), and the third one of (6) (or (5): if f is uniformly Lipschitz, then it satisfies $f(v) \leq C_0(1+|v|)$, and thus the condition in the third point of the lemma is satisfied). Hence this example somehow explains the different conditions which are required for existence, and enlights the various regimes which can occur. However, hypotheses (5), (6) and (7) do not cover all the cases in which the existence holds, even in this rather simplified problem. A more general and more thorough existence theory remains to be accomplished.

As a conclusion to this subsection, let us also mention that the above example also provides cases when the convergences (20), (21) do not hold. Indeed, assume that $\alpha, \beta \in \mathbb{R}$ and that solutions of (8) (or, equivalently, of (32)) exist for all $p \in \mathbb{R}$. Then

$$\lim_{p \to +\infty} \inf_Y v(y,p) = \lim_{c \to c_+} H^{-1}(\inf \phi + c) = H^{-1}(\beta - \mathrm{osc}\phi) < +\infty$$

and similarly $\lim_{p \to -\infty} \sup_Y v(y,p) > -\infty$.

2.2 Evolution equation and first order corrector

Once $\tilde{u}$ is rigourously defined, we can compute the homogenized flux

$$\bar{A}(p) := \langle A(\cdot, p + \tilde{u}(\cdot, p)) \rangle \tag{35}$$

Define also for $1 \le i \le N$

$$\bar{a}_i(p) := \frac{\partial \bar{A}_i(p)}{\partial p} = \left\langle \frac{\partial v(\cdot,p)}{\partial p}\; a_i(\cdot, v(\cdot,p)) \right\rangle .$$

Then according to the results of the preceding subsection, $\bar{a}_i \in L^\infty_{\rm loc}(\mathbb{R})$.

$\bar{u}$ can thus be defined as the unique entropy solution in $\mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty)\times \mathbb{R}^N)$ of the scalar conservation law (see for instance [15] for a complete theory of existence and uniqueness)

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \displaystyle\sum_{i=1}^N \frac{\partial \bar{A}_i(\bar{u})}{\partial x_i} = 0, \quad t \ge 0,\ x \in \mathbb{R}^N \\ \bar{u}(t=0,x) = \bar{u}_0(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \end{cases} \tag{36}$$

On sets of $[0,\infty)\times\mathbb{R}^N$ on which $\bar{u}$ is regular (say $W^{1,1}$), one can define the first order corrector u^1 by

$$-\Delta_y u^1 + \sum_{i=1}^N \frac{\partial}{\partial y_i}\left[u^1 a_i(y,u^0)\right] = 2\Delta_{xy}u^0 - \left\{\frac{\partial u^0}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i}\left[A_i(y,u^0)\right]\right\}. \tag{37}$$

(t,x) are parameters; since the right hand side has mean zero, and the only solutions of the adjoint equation

$$-\Delta_y w - a_i(y,u^0)\cdot\nabla_y w = 0$$

are constants, one can apply the Riesz-Fredholm theory to show that solutions of (37) exist, and are unique up to solutions of the homogeneous equation

$$-\Delta_y w + \sum_{i=1}^N \frac{\partial}{\partial y_i}\left[w a_i(y,u^0)\right] = 0.$$

Comparing the above equation to (17), and recalling the results of the proof of lemma 6, we see that the solutions of the homogeneous equation can be written $w(t,x,y) = c(t,x)\frac{\partial v}{\partial p}(y,\bar{u}(t,x))$. In particular, u^1 is unequivocally defined under the condition

$$\int_Y u^1(t,x,y)\frac{\partial v}{\partial p}(t,x,y)\,dy = 0 \quad \text{a.e. } (t,x)\in[0,\infty)\times\mathbb{R}^N.$$

Pushing the calculations a little further, we write u^1 in the slightly more sympathetic form

$$u^1(t,x,y) = \sum_{i=1}^N \frac{\partial \bar{u}(t,x)}{\partial x_i}\chi_i(y,\bar{u}(t,x)),$$

where $\chi_i(\cdot,p)\in H^1_{\rm per}(Y)\ \forall p\in\mathbb{R}$ solves the elliptic equation :

$$-\Delta_y \chi_i + \sum_{j=1}^{N} \frac{\partial}{\partial y_j} \left(a_j(y, v(y,p)) \chi_i \right) = 2 \frac{\partial^2 v(y,p)}{\partial y_i \, \partial p} + \frac{\partial v(y,p)}{\partial p} \frac{\partial}{\partial p} \left\langle A_i(\cdot, v(\cdot, p)) \right\rangle_Y - \frac{\partial}{\partial p} \left(A_i(y, v(y,p)) \right). \tag{38}$$

As before, the existence and uniqueness of χ_i follow from the Fredholm alternative provided the condition

$$\int_Y \chi_i \frac{\partial v}{\partial p} = 0 \quad \forall p \in \mathbb{R}$$

holds true.

Let us summarize the results of this subsection in the following

Lemma 14 *Assume $A \in W^{1,\infty}_{per,\, loc}(Y \times \mathbb{R})$ satisfies (3), (4). Then there exists a unique entropy solution $\bar{u} \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty) \times \mathbb{R}^N)$ of the hyperbolic scalar conservation law (36).*

If $\bar{u} \in W^{1,1}(\mathcal{O})$, where $\mathcal{O} \subset [0,\infty) \times \mathbb{R}^N$, then for $(t,x) \in \mathcal{O}$, there exists a unique $u^1(t,x,\cdot) \in H^1_{per}(Y)$ satisfying (37) and the condition

$$\int_Y u^1(t,x,y) \frac{\partial v}{\partial p}(t,x,y) \, dy = 0 \quad a.e. \ (t,x) \in \mathcal{O}.$$

Moreover, u^1 can be written

$$u^1(t,x,y) = \sum_{i=1}^{N} \frac{\partial \bar{u}(t,x)}{\partial x_i} \chi_i(y, \bar{u}(t,x)),$$

where $\chi_i(\cdot, p) \in H^1_{per}(Y)$ satisfies equation (38) $\forall p \in \mathbb{R}$.

In the rest of the article, we set

$$u^0(t,x,y) := \bar{u}(t,x) + \tilde{u}(y, \bar{u}(t,x)) = v(y, \bar{u}(t,x)). \tag{39}$$

3 Convergence proof

3.1 Naive idea using L^1 contraction principle

We are now ready to prove the convergence result announced in theorem 3. A first naive idea consists in computing the equation satisfied by $u^0\left(t,x,\frac{x}{\varepsilon}\right)$, or rather

$$v^\varepsilon(t,x) := u^0\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t,x,\frac{x}{\varepsilon}\right),$$

where u^0 and u^1 were defined in the last section : assuming that $\bar{u}$ and A are regular in order to compute all the necessary derivations, v^ε is a solution of

$$\frac{\partial v^\varepsilon}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i \left(\frac{x}{\varepsilon}, v^\varepsilon\right) - \varepsilon \Delta_x v^\varepsilon = f^\varepsilon,$$

where

$$\begin{aligned} f^\varepsilon(t,x) = &\frac{1}{\varepsilon}\left\{\frac{\partial A_i}{\partial y_i}\left(\frac{x}{\varepsilon}, u^0+\varepsilon u^1\right) - \frac{\partial A_i}{\partial y_i}\left(\frac{x}{\varepsilon}, u^0\right) - \varepsilon u^1 \frac{\partial^2 A_i}{\partial y_i\,\partial v}\left(\frac{x}{\varepsilon}, u^0\right)\right\} \\ &+ \frac{1}{\varepsilon}\left\{\frac{\partial u^0}{\partial y_i}\left[\frac{\partial A_i}{\partial v}\left(\frac{x}{\varepsilon}, u^0+\varepsilon u^1\right) - \frac{\partial A_i}{\partial v}\left(\frac{x}{\varepsilon}, u^0\right) - \varepsilon u^1 \frac{\partial^2 A_i}{\partial v^2}\left(\frac{x}{\varepsilon}, u^0\right)\right]\right\} \\ &+ \varepsilon^0\left\{\left(\frac{\partial u^1}{\partial y_i} + \frac{\partial u^0}{\partial x_i}\right)\left[\frac{\partial A_i}{\partial v}\left(\frac{x}{\varepsilon}, u^0+\varepsilon u^1\right) - \frac{\partial A_i}{\partial v}\left(\frac{x}{\varepsilon}, u^0\right)\right]\right\} \\ &+ \varepsilon^1\left\{\frac{\partial u^1}{\partial t} + \frac{\partial u^1}{\partial x_i} a_i\left(\frac{x}{\varepsilon}, u^0+\varepsilon u^1\right) - \Delta_x u^0 - 2\Delta_{xy} u^1\right\} \\ &- \varepsilon^2 \Delta_x u^1. \end{aligned} \tag{40}$$

Assuming that u^ε satisfies (11),

$$v^\varepsilon(t=0,x) - u^\varepsilon(t=0,x) = \varepsilon u^1\left(t=0, x, \frac{x}{\varepsilon}\right) = \varepsilon \sum_{i=1}^N \frac{\partial \bar{u}}{\partial x_i}\chi_i\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right).$$

We assume that $a_{N+1}(y,0) = 0$, so that $u^\varepsilon(t)$ and $v^\varepsilon(t)$ belong to $L^1(\mathbb{R}^N)$ for all $t \geq 0$. Thus, using the L^1 contraction property for equation (1) yields :

$$||u^\varepsilon(T) - v^\varepsilon(T)||_{L^1(\mathbb{R}^N)} \leq \varepsilon ||u^1\left(t=0, x, \frac{x}{\varepsilon}\right)||_{L^1(\mathbb{R}^N)} + \int_0^T \int_{\mathbb{R}^N} |f^\varepsilon(t,x)|\,dx\,dt.$$

The next step consists in deriving a bound of order ε for f^ε. The calculations are lengthy and fastidious, and require very strong regularity assumptions on $\bar{u}$ and on the flux A : for instance, in order to upper bound the first two terms in (40), which are Taylor expansions, we need to assume $A \in W^{3,\infty}_{\text{per}}(Y \times \mathbb{R})^N$. Eventually, we obtain the following rough estimate :

$$\int_0^T \int_{\mathbb{R}^N} |f^\varepsilon(t,x)|\,dx\,dt \leq C\varepsilon \int_0^T \int_{\mathbb{R}^N} g^\varepsilon(t,x)\,dx\,dt,$$

where

$$g^\varepsilon(t,x) := |\nabla_x \bar{u}|^2 + |\partial_t \bar{u}||\nabla_x \bar{u}| + |D^2\bar{u}| + |\partial_t \nabla_x \bar{u}| + |D^3\bar{u}| + |\nabla_x \bar{u}|^3 + |D^2\bar{u}||\nabla_x \bar{u}| \tag{41}$$

and C is a constant depending only on N, Y, and the bounds on A.

We do not give the details of the proof here; the main advantage of this method is to give a better understanding of the problem thanks to explicit calculations. The proof we will give of theorem 3 in this article does not require as many calculations, but might seem less intuitive since the convergence is “hidden” behind Young measures.

3.2 A few results about two-scale Young measures

Let us first recall a few results about two-scale Young measures : standard Young measures were introduced by Luc Tartar in [16] in the framework of compensated compactness as a tool to study weak limits of non-linear functions. Weinan E in [4] combined Tartar's results with Nguetseng's and Allaire's theory of two-scale convergence (see [1], [7]) and proved the following lemma:

Lemma 15 *Assume we have a sequence of functions $\{v^\varepsilon\}_{\varepsilon>0}$, with $v^\varepsilon : \mathbb{R}^N \to K$, where K is a compact set of $\mathbb{R}$. Then there exists a subsequence, still denoted by $\{v^\varepsilon\}_{\varepsilon>0}$, and a family of parametrized probability measures $\{\nu_{x,y}(\lambda)\}$ supported in K, which depends measurably on (x, y), and is periodic in y with period Y, such that as $\varepsilon \to 0$,*

$$\int_{\mathbb{R}^N} F(v^\varepsilon(t,x))\psi\left(x, \frac{x}{\varepsilon}\right)\, dx \to \int_{\mathbb{R}^N\times Y} \langle F(\lambda), \nu_{x,y}\rangle\, \psi(x,y)\, dx\, dy \tag{42}$$

for all $\psi \in K_{per}$, $F \in \mathcal{C}(K)$. The subsequence does not depend on ψ or F.

$\{\nu_{x,y}(\lambda)\}$ is the two-scale Young measure associated to the sequence v^ε.

For our application, we will need the following straightforward generalization of E's lemma :

Corollary 16 *Assume we have a sequence of functions $\{v^\varepsilon\}_{\varepsilon>0}$, with $v^\varepsilon : [0,\infty) \times \mathbb{R}^N \to K$, where K is a compact set of $\mathbb{R}$. Then there exists a subsequence, still denoted by $\{v^\varepsilon\}_{\varepsilon>0}$, and a family of parametrized probability measures $\{\nu_{t,x,y}(\lambda)\}$ supported in K, which depends measurably on (t, x, y), and is periodic in y with period Y, such that as $\varepsilon \to 0$,*

$$\int_{[0,\infty)\times\mathbb{R}^N} F\left(\frac{x}{\varepsilon}, v^\varepsilon(t,x)\right)\psi\left(t, x, \frac{x}{\varepsilon}\right)\, dtdx \to \int_{[0,\infty)\times\mathbb{R}^N\times Y} \langle F(y,\lambda), \nu_{t,x,y}\rangle\, \psi(t,x,y)\, dtdxdy \tag{43}$$

for all $\psi \in J_{per}$, $F \in \mathcal{C}_{per}(Y \times K)$. The subsequence does not depend on ψ or F.

We will also use the following lemma, due to Tartar (see [16]):

Lemma 17 *The two-scale Young measure $\{\nu_{t,x,y}\}$ associated with $\{v^\varepsilon\}_{\varepsilon>0}$ reduces to a family of Dirac measures $\delta_{V(t,x,y)}$ if and only if*

$$\left\| v^\varepsilon(t,x) - V\left(t, x, \frac{x}{\varepsilon}\right)\right\|_{L^2_{loc}([0,\infty)\times\mathbb{R}^n)} \to 0.$$

We want to apply corollary 16 to the sequence u^ε of solutions of (1). Let us prove that u^ε is bounded. First, recall that $u^\varepsilon(t=0,x) = v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)$ with $\bar{u}_0 \in L^\infty(\mathbb{R}^N)$. Thus, setting $C = ||\bar{u}_0||_{L^\infty(\mathbb{R}^N)}$ and recalling (18), we have

$$v\left(\frac{x}{\varepsilon}, -C\right) \le u^\varepsilon(t=0,x) \le v\left(\frac{x}{\varepsilon}, C\right) \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

Then, notice that for all $p \in \mathbb{R}$, $v\left(\frac{x}{\varepsilon}, p\right)$ is a stationary solution of (1) and that the evolution operator associated to (1) is order preserving. Hence,

$$v\left(\frac{x}{\varepsilon}, -C\right) \le u^\varepsilon(t,x) \le v\left(\frac{x}{\varepsilon}, C\right) \qquad \text{for a.e. } t,x \in [0,\infty) \times \mathbb{R}^N$$

and

$$||u^\varepsilon||_{L^\infty([0,\infty)\times\mathbb{R}^N)} \le \max\left(||v(\cdot, C)||_{L^\infty(Y)}, ||v(\cdot,-C)||_{L^\infty(Y)}\right) =: k.$$

Thanks to this estimate, we can use corollary 16 for the sequence u^ε, with $K = [-k,k]$. Let $\nu_{t,x,y}$ be the two-scale Young measure associated to the sequence u^ε. As in [4], [3], [5], the goal is to reduce the family $\{\nu_{t,x,y}\}_{t,x,y}$ to a family of Dirac masses, which will lead to the strong convergence in L^2_{loc}, as announced in theorem 3.

3.3 Formulation of the cell problem in terms of Young measures

With this aim in view, we use once again the fact that $v\left(\frac{x}{\varepsilon}, p\right)$ is a stationary solution of (1), combined with the L^1 contraction principle for equation (1) and we obtain the following inequality:

$$\frac{\partial}{\partial t}\left|u^\varepsilon - v\left(\frac{x}{\varepsilon}, p\right)\right| + \sum_{i=1}^N \frac{\partial}{\partial x_i}\left[\text{sgn}\left(u^\varepsilon - v\left(\frac{x}{\varepsilon}, p\right)\right)\left(A_i\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - A_i\left(\frac{x}{\varepsilon}, v\left(\frac{x}{\varepsilon}, p\right)\right)\right)\right] - \varepsilon\Delta_x\left|u^\varepsilon - v\left(\frac{x}{\varepsilon}, p\right)\right| \le 0 \quad (44)$$

In a first step, we multiply (44) by positive test functions $\varepsilon\varphi\left(t,x,\frac{x}{\varepsilon}\right)$, where $\varphi \in J_{\text{per}}$, and we pass to the limit as $\varepsilon \to 0$ using corollary 16 in order to derive information at the microscopic level on the mesure ν. This leads to the inequality (in the sense of distributions on $[0,\infty) \times \mathbb{R}^N \times Y$, for all $p \in \mathbb{R}$) :

$$-\Delta_y \langle |\lambda - v(y,p)|, \nu_{t,x,y}\rangle + \text{div}_y \langle \text{sgn}(\lambda - v(y,p))[A(y,\lambda) - A(y, v(y,p))], \nu_{t,x,y}\rangle \le 0.$$

Since the left-hand side has mean zero, the inequality is in fact an equality :

$$\Delta_y \langle |\lambda - v(y,p)|, \nu_{t,x,y}\rangle + \text{div}_y \langle \text{sgn}(\lambda - v(y,p))[A(y,\lambda) - A(y, v(y,p))], \nu_{t,x,y}\rangle = 0. \quad (45)$$

As we shall see in the sequel of the proof, we need to prove that $\langle \text{sgn}(\lambda - v(y,p)), \nu_{t,x,y}\rangle$ is well defined and independant of $y \in Y$. This result can be obtained in a rather simple and straightforward way by deriving equation (45) with respect to p; unfortunately, this manipulation is valid if and only if $\nu_{t,x,y}(v(y,p)) = 0$. However, deriving equation (45) on the right and on the left yields the following lemma :

Lemma 18 *We use the convention*

$$\langle \operatorname{sgn}(\lambda - \alpha), \nu_{t,x,y} \rangle := \nu_{t,x,y}(\lambda > \alpha) - \nu_{t,x,y}(\lambda < \alpha).$$

Then for all $p \in \mathbb{R}$, $\langle \operatorname{sgn}(\lambda - v(y,p)), \nu_{t,x,y} \rangle$ is well defined and is independant of $y \in Y$: there exists $C = C(t,x,p) \in L^\infty([0,\infty) \times \mathbb{R}^N \times \mathbb{R})$ such that

$$\langle \operatorname{sgn}(\lambda - v(y,p)), \nu_{t,x,y} \rangle = C(t,x,p) \quad \textit{for a.e. } (t,x,y) \in [0,\infty) \times \mathbb{R}^N \times Y, \ \forall p \in \mathbb{R}.$$

We postpone the proof of the lemma to subsection 3.5.

3.4 Reduction of Young measures

As in [4], [5], [3], we apply DiPerna's method in [17] to reduce the family $\{\nu_{t,x,y}\}_{t,x,y}$ to a family of Dirac masses : we want to prove that

$$\partial_t \int_Y \left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y} \right\rangle \, dy + \partial_{x_i} \int_Y \left\langle \eta_i(y, \lambda, u^0(t,x,y)), \nu_{t,x,y} \right\rangle \, dy \le 0, \tag{46}$$

where

$$\eta_i(y,\lambda,v) := \operatorname{sgn}(\lambda - v) \left[A_i(y,\lambda) - A_i(y,v) \right].$$

Indeed, if (46) is true, then we multiply (46) by $e^{-|x|}$ (recall that u^0 is bounded in L^∞, but not in $L^1(\mathbb{R}^N \times Y)$ in general) and we get

$$\frac{d}{dt} \int_{\mathbb{R}^N \times Y} \left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y} \right\rangle e^{-|x|} \, dydx \le C \int_{\mathbb{R}^N \times Y} \left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y} \right\rangle e^{-|x|} \, dydx,$$

where $C = ||a_i||_{L^\infty(Y \times [-k,k])}$. Hence, by Gronwall's lemma,

$$\int_{\mathbb{R}^N \times Y} \left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y} \right\rangle e^{-|x|} \, dydx \le e^{Ct} \int_{\mathbb{R}^N \times Y} \langle |\lambda - v(y, \bar{u}_0(x))|, \nu_{t=0,x,y} \rangle \, e^{-|x|} \, dydx. \tag{47}$$

Moreover, since the initial data is well prepared thanks to (11),

$$\left\langle |\lambda - u^0(t=0,x,y)|, \nu_{t=0,x,y} \right\rangle = 0. \tag{48}$$

Thus, combining (47) and (48), we obtain

$$\left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y} \right\rangle = 0 \quad \text{for a.e. } (t,x,y) \in [0,\infty) \times \mathbb{R}^N \times Y,$$

which entails

$$\nu_{t,x,y} = \delta_{u^0(t,x,y)}. \tag{49}$$

(46) remains to be proved. Formally, the left-hand side of (46) can be split into a sum of two terms :

$$\int_Y \left[\left\langle |\lambda - u^0(t,x,y)|, \partial_t \nu_{t,x,y} \right\rangle + \sum_{i=1}^N \left\langle \eta_i(y, \lambda, u^0(t,x,y)), \partial_{x_i} \nu_{t,x,y} \right\rangle \right] dy \tag{50}$$

and

$$\int_Y \left\langle \partial_t |\lambda - u^0(t,x,y)| + \sum_{i=1}^N \partial_{x_i} \eta_i(y, \lambda, u^0(t,x,y)), \nu_{t,x,y} \right\rangle dy. \tag{51}$$

First, in order to prove that (50) is nonpositive, we multiply (44) by nonnegative test functions $\varphi = \varphi(t,x) \in \mathcal{D}([0,\infty) \times \mathbb{R}^N)_+$ and pass to the limit as $\varepsilon \to 0$ using once again corollary 16. We obtain, in the sense of distributions and for all $p \in \mathbb{R}$,

$$\partial_t \int_Y \langle |\lambda - v(y,p)|, \nu_{t,x,y} \rangle + \sum_{i=1}^N \partial_{x_i} \int_Y \langle \eta_i(y, \lambda, v(y,p)), \nu \rangle \leq 0. \tag{52}$$

(52) yields

$$\int_Y \left[\langle |\lambda - v(y,p)|, \partial_t \nu_{t,x,y} \rangle + \sum_{i=1}^N \langle \eta_i(y, \lambda, v(y,p)), \partial_{x_i} \nu_{t,x,y} \rangle \right] dy \leq 0$$

for all $p \in \mathbb{R}$. The choice $p = \bar{u}(t,x)$ implies that (50) is nonpositive.

Proving that the term (51) is nonpositive is a bit more difficult, mainly because if $\bar{u}$ is an entropy solution of (36), there is no reason why u^0 should be an entropy solution of the scalar law

$$\frac{\partial u^0}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i(y, u^0) = g(t,x,y),$$

where g is a source term with null Y-average (recall that u^1 is defined only on the sets on which $\bar{u}$ is regular; on such sets, it can be proved that u^0 is indeed an entropy solution of such a law).

The idea is to use the results on kinetic formulation of conservation laws (see for instance [15]): if $S \in \mathcal{C}^2(\mathbb{R})$, then

$$\frac{\partial S(\bar{u})}{\partial t} + \sum_{i=1}^N \frac{\partial \bar{\eta}_i(\bar{u})}{\partial x_i} = -\int_{\mathbb{R}} S''(p) m(t,x,p)\, dp, \tag{53}$$

where m is the entropy defect measure associated to $\bar{u}$, and $\bar{\eta}_i$ is defined by

$$\bar{\eta}_i'(p) = \bar{a}_i(p) S'(p).$$

Set, for $(y,\lambda)\in Y\times\mathbb{R}$,

$$S^{y,\lambda}(p) := |v(y,p)-\lambda|,$$
$$\eta_i^{y,\lambda}(p) := \int_0^p \bar{a}_i(q)\mathrm{sgn}(v(y,q)-\lambda)\frac{\partial v(y,q)}{\partial q}\, dq.$$

Unfortunately $S^{y,\lambda}$ is not $\mathcal{C}^2$: thus, we use (53) for $S^{y,\lambda,\delta}(p) := S^{y,\lambda} * \varphi_\delta(p)$, where φ_δ is a standard mollifier, and we let $\delta \to 0$. It can be readily shown that $S^{y,\lambda,\delta}$ (resp. $\eta_i^{y,\lambda,\delta}$) converges to $S^{y,\lambda}$ (resp. $\eta_i^{y,\lambda}$) uniformly on compact sets of $\mathbb{R}$ and uniformly for $(y,\lambda)\in Y\times K$ (recall that $\nu_{t,x,y}$ is supported in K). Thus as $\delta\to 0$, in the sense of distributions on $[0,\infty)\times\mathbb{R}^N$,

$$\int_Y \left\langle \partial_t S^{y,\lambda,\delta}(\bar{u}(t,x)), \nu_{t,x,y}\right\rangle\, dy \rightharpoonup \int_Y \left\langle \partial_t S^{y,\lambda}(\bar{u}(t,x)), \nu_{t,x,y}\right\rangle\, dy,$$

and the same convergence holds for $\partial_{x_i}\eta_i^{y,\lambda,\delta}(\bar{u})$.

On the other hand,

$$S^{y,\lambda,\delta\prime\prime}(p) = \int_{\mathbb{R}} \frac{\partial v}{\partial p}(y,p')\mathrm{sgn}(v(y,p')-\lambda)\varphi_\delta'(p-p')\, dp';$$

using lemma 18 and the property $\left\langle \frac{\partial v}{\partial p}\right\rangle = 1$ yields

$$\int_Y \left\langle S^{y,\lambda,\delta\prime\prime}(p), \nu_{t,x,y}\right\rangle\, dy = \int_Y \left\langle \left[\int_{\mathbb{R}} \mathrm{sgn}(v(y,p')-\lambda)\varphi_\delta'(p-p')\, dp'\right], \nu_{t,x,y}\right\rangle\, dy.$$

Then, using a regularization of the function signum it can be proved that

$$\int_{\mathbb{R}} \mathrm{sgn}(v(y,p')-\lambda)\varphi_\delta'(p-p')\, dp' \geq 0 \quad \forall y\in Y, \lambda\in\mathbb{R},$$

and consequently

$$-\int_{\mathbb{R}\times Y} \left\langle S^{y,\lambda,\delta\prime\prime}(p), \nu_{t,x,y}\right\rangle\, m(t,x,p)\, dp\, dy \leq 0.$$

Thus, passing to the limit as $\delta\to 0$, we obtain

$$\int_Y \left\langle \partial_t|\lambda - v(y,\bar{u}(t,x))| + \sum_{i=1}^N \partial_{x_i}\eta_i^{y,\lambda}(\bar{u}), \nu_{t,x,y}\right\rangle\, dy \leq 0 \tag{54}$$

where the inequality is meant in the sense of distributions.

We split (51) into

$$
\begin{aligned}
&\int_Y \left\langle \partial_t |\lambda - u^0(t,x,y)| + \sum_{i=1}^N \partial_{x_i} \eta_i(y,\lambda,u^0(t,x,y)), \nu_{t,x,y} \right\rangle \, dy \\
=&\int_Y \left\langle \partial_t |\lambda - u^0(t,x,y)| + \sum_{i=1}^N \partial_{x_i} \eta_i^{y,\lambda}(\bar{u}), \nu_{t,x,y} \right\rangle \, dy \qquad (55)\\
&+\sum_{i=1}^N \int_Y \left\langle \partial_{x_i} \left[\eta_i(y,\lambda,u^0(t,x,y)) - \eta_i^{y,\lambda}(\bar{u}) \right], \nu_{t,x,y} \right\rangle \, dy \qquad (56)
\end{aligned}
$$

Thanks to (54), (55) is nonpositive. Let us now focus on (56) : set

$$
\begin{aligned}
f^i(y,\lambda,p) :=& \frac{\partial}{\partial p} \left[\eta_i(y,\lambda,v(y,p)) - \eta_i^{y,\lambda}(p) \right] \\
=& \operatorname{sgn}(v(y,p)-\lambda) \frac{\partial v}{\partial p} \left[a_i(y,v(y,p)) - \bar{a}_i(p) \right]
\end{aligned}
$$

Using once again lemma 18 and the definition of $\bar{a}_i$ yields

$$
\int_Y \left\langle f^i(y,\lambda,p), \nu_{t,x,y} \right\rangle \, dy = 0 \quad \forall p \in \mathbb{R}. \qquad (57)
$$

Set

$$
F^i(y,\lambda,q) := \int_0^q f^i(y,\lambda,p) \, dp;
$$

Then (56) is equal to

$$
\begin{aligned}
&\sum_{i=1}^N \int_Y \left\langle \partial_{x_i} F^i(y,\lambda,\bar{u}(t,x)), \nu_{t,x,y} \right\rangle \, dy \\
=&\sum_{i=1}^N \partial_{x_i} \int_Y \left\langle F^i(y,\lambda,\bar{u}(t,x)), \nu_{t,x,y} \right\rangle \, dy - \int_Y \left\langle F^i(y,\lambda,\bar{u}(t,x)), \partial_{x_i} \nu_{t,x,y} \right\rangle \, dy.
\end{aligned}
$$

(57) entails that

$$
\int_Y \left\langle F^i(y,\lambda,q), \nu_{t,x,y} \right\rangle \, dy = 0 \qquad \text{for a.e. } (t,x) \in [0,\infty) \times \mathbb{R}^N \ \forall q \in \mathbb{R},
$$

and thus (56) is null as well. Hence, we have proved (46), and the family $\{\nu_{t,x,y}\}_{t,x,y}$ is reduced to a family of Dirac masses.

□

Remark 19 *In fact, several regularizations are necessary in order to make the proof rigorous; for instance, we need to regularize the measure ν with respect to t, x, so that the quantities $\partial_t \nu$, $\partial_{x_i} \nu$ are well-defined and the properties of lemma 18 are preserved, together with inequality* (52)*. These calculations are straight-forward and follow the arguments developed by R. DiPerna in [17].*

Let us stress as well that the equality $\nu_{t=0,x,y} = \delta_{u_0(x,y)}$ is not obvious: indeed, uniform bounds in ε on $\left\|u^\varepsilon(t) - u_0\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^1(\mathbb{R}^N)}$, for t close to 0, are not easy to derive; a simple way to prove this fact is to go back to inequality (44), which yields:

$$\limsup_{t\to 0} \int_{\mathbb{R}^N\times Y} \langle|\lambda - v(y,p)|, \nu_{t,x,y}\rangle\, \varphi(x)\, dxdy \le \int_{\mathbb{R}^N\times Y} |v(y, \bar{u}_0(x)) - v(y,p)|\, \varphi(x)\, dxdy,$$

for all $p \in \mathbb{R}$, $\varphi \in \mathcal{D}(\mathbb{R}^N)_+$. Hence, for any measure $\mu_{x,y}(\lambda)$ such that there exists a sequence $t_n \to 0$ with $\nu_{t_n,x,y} \rightharpoonup \mu_{x,y}$ w-$M^1(\mathbb{R}_\lambda \times \mathbb{R}^N \times Y)$, we have

$$\int_Y \langle|\lambda - v(y,p)|, \mu_{x,y}\rangle\; dy \le \int_Y |v(y, \bar{u}_0(x)) - v(y,p)|\; dy$$

for all $p \in \mathbb{R}$ and in the sense of measures for $x \in \mathbb{R}^N$. Taking $p = \bar{u}_0(x)$ gives $\mu_{x,y} = \delta_{u_0(x,y)}$, and thus the whole sequence $\nu_{t,x,y}$ converges in w-M^1 to $\delta_{u_0(x,y)}$ as $t \to 0$.

3.5 Proof of lemma 18

First, observe that $\langle|\lambda - v(y,p)|, \nu_{t,x,y}\rangle$ is a continuous function of p for a.e. $(t,x,y) \in [0,\infty)\times\mathbb{R}^N\times Y$. Moreover, if $\lambda \ne v(y,p_0)$, then the function $f_y(\lambda,p) := |\lambda - v(y,p)|$ has a partial derivative with respect to p at the point (λ, p_0) which is equal to

$$\frac{\partial f_y}{\partial p}(\lambda, p_0) = -\frac{\partial v}{\partial p}(y,p_0)\mathrm{sgn}(\lambda - v(y,p_0)).$$

If $\lambda = v(y,p_0)$, then f_y has a partial derivatives on the right and on the left at the point (λ, p_0) which are equal to

$$\begin{aligned} \frac{\partial f_y}{\partial p}(v(y,p_0), p_0^+) &= \frac{\partial v}{\partial p}(y,p_0), \\ \frac{\partial f_y}{\partial p}(v(y,p_0), p_0^-) &= -\frac{\partial v}{\partial p}(y,p_0). \end{aligned}$$

Additionnally, notice that for all $\lambda \in \mathbb{R}$, $p \ne p_0$

$$\left|\frac{f_y(\lambda,p) - f_y(\lambda,p_0)}{p-p_0}\right| \le \left|\frac{v(y,p) - v(y,p_0)}{p-p_0}\right| \le \left\|\frac{\partial v}{\partial p}\right\|_{L^\infty(Y\times\mathbb{R})}.$$

Hence, using Lebesgue's dominated convergence theorem, we deduce that the function

$$F_{t,x,y}(p) := \langle|\lambda - v(y,p)|, \nu_{t,x,y}\rangle$$

has derivatives on the right and on the left with respect to p for almost every $(t,x,y)\in[0,\infty)\times\mathbb{R}^N\times Y$:

$$F'_{t,x,y}(p_0^+) = -\frac{\partial v}{\partial p}(y,p_0)\,\langle \operatorname{sgn}(\lambda - v(y,p_0)),\nu_{t,x,y}\rangle + \nu_{t,x,y}\left(\{v(y,p_0)\}\right)\frac{\partial v}{\partial p}(y,p_0),$$
$$F'_{t,x,y}(p_0^-) = -\frac{\partial v}{\partial p}(y,p_0)\,\langle \operatorname{sgn}(\lambda - v(y,p_0)),\nu_{t,x,y}\rangle - \nu_{t,x,y}\left(\{v(y,p_0)\}\right)\frac{\partial v}{\partial p}(y,p_0).$$

In a similar fashion, the function $G^i_{t,x,y}(p) := \langle \operatorname{sgn}(\lambda - v(y,p))\left[A_i(y,\lambda) - A_i(y,v(y,p))\right],\nu_{t,x,y}\rangle$ has derivatives on the right and on the left with respect to p at $p=p_0$ which are equal to

$$G^{i\prime}_{t,x,y}(p_0^+) = -\frac{\partial v}{\partial p}(y,p_0)a_i(y,v(y,p_0))\left[\langle \operatorname{sgn}(\lambda - v(y,p_0)),\nu_{t,x,y}\rangle - \nu_{t,x,y}\left(\{v(y,p_0)\}\right)\right],$$
$$G^{i\prime}_{t,x,y}(p_0^-) = -\frac{\partial v}{\partial p}(y,p_0)a_i(y,v(y,p_0))\left[\langle \operatorname{sgn}(\lambda - v(y,p_0)),\nu_{t,x,y}\rangle + \nu_{t,x,y}\left(\{v(y,p_0)\}\right)\right].$$

Thus, setting

$$r(t,x,y,p) := \frac{\partial v}{\partial p}(y,p)\left[\langle \operatorname{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\rangle - \nu_{t,x,y}\left(\{v(y,p)\}\right)\right],$$
$$l(t,x,y,p) := \frac{\partial v}{\partial p}(y,p)\left[\langle \operatorname{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\rangle + \nu_{t,x,y}\left(\{v(y,p)\}\right)\right],$$

we see that l and r both satisfy for all $p\in\mathbb{R}$ the elliptic equation

$$-\Delta_y g + \operatorname{div}_y\left(a(y,v(y,p))g\right) = 0 \tag{58}$$

a.e. on $[0,\infty)\times\mathbb{R}^N$ and in the sense of distributions on Y. Thus $l,r\in H^1_{\text{per}}(Y)$ for all $p\in\mathbb{R}$ and for a.e. $(t,x)\in[0,\infty)\times\mathbb{R}^N$, and the equation is satisfied in the variational sense for elliptic equations.

Comparing (58) to (17), and using the Krein-Rutman theorem (see lemma 6), we deduce that there exist constants $C_r = C_r(t,x,p)$ and $C_l = C_l(t,x,p)$ such that

$$r(t,x,y,p) = C_r(t,x,p)\frac{\partial v}{\partial p}(y,p),$$
$$l(t,x,y,p) = C_l(t,x,p)\frac{\partial v}{\partial p}(y,p)$$

Since $\frac{\partial v}{\partial p}$ is a positive function which does not vanish on Y (see lemma 6), this yields

$$\langle \operatorname{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\rangle - \nu_{t,x,y}\left(\{v(y,p)\}\right) = C_r(t,x,p),$$
$$\langle \operatorname{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\rangle + \nu_{t,x,y}\left(\{v(y,p)\}\right) = C_l(t,x,p).$$

Thus,

$$\langle \operatorname{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\rangle = \frac{1}{2}\left(C_l(t,x,p) + C_r(t,x,p)\right) = C(t,x,p).$$

and the proof is complete. Notice that we have also proved that $\nu_{t,x,y}\left(\{v(y,p)\}\right)$ does not depend on y.

□

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