Homogenization of linear transport equations in a stationary ergodic setting

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Abstract

We study the homogenization of a linear kinetic equation which models the evolution of the density of charged particles submitted to a highly oscillating electric field. The electric field and the initial density are assumed to be random and stationary. We identify the asymptotic microscopic and macroscopic profiles of the density, and we derive formulas for these profiles when the space dimension is equal to one.

1 Introduction

This note is concerned with the homogenization of a linear transport equation in a stationary ergodic setting. The equation studied here describes the evolution of the density of charged particles in a rapidly oscillating random electric potential. This equation can be derived by passing to the semi-classical limit in the Schrödinger equation (see [12], [14], and the presentation in [10]). Our work generalizes a result of E. Frénod and K. Hamdache (see [10]) which was obtained in a periodic setting. The strategy of proof we have chosen here is different from the one of [10], and allows us to retrieve some of the results in [10] in a rather simple and explicit fashion.

Let us mention a few related works on the homogenization of linear transport equations ; we emphasize that this list is by no means exhaustive. In [2], Y. Amirat, K. Hamdache and A. Ziani study the homogenization of a linear transport equation in a periodic setting and give an application to a model describing a multidimensional miscible flow in a porous media. In [6] (see also [11]), Laurent Dumas and François Golse focus on the homogenization of linear transport equations with absorption and scattering terms, in periodic and stationary ergodic settings. And in [7], Weinan E derives strong convergence results for the homogenization of linear and nonlinear transport equations with oscillatory incompressible velocity fields in a periodic setting.

Let us now present the context we will be working in : let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(\tau_x)_{x\in\mathbb{R}^N}$ be a group transformation acting on Ω. We assume that τ_x preserves the probability measure P for all $x \in \mathbb{R}^N$, and the group transformation is ergodic, which means

$$\forall A \in \mathcal{F}, \quad \left(\tau_x A = A \ \forall x \in \mathbb{R}^N \Rightarrow P(A) = 0 \text{ or } 1\right).$$

The periodic setting can be embedded the stationary ergodic setting in the following way (see [17]) : take $\Omega = \mathbb{R}^N/\mathbb{Z}^N \simeq [0,1)^N$, and let P be the Lebesgue measure on Ω. Define the group transformation $(\tau_x)_{x\in\mathbb{R}^N}$ by

$$\tau_x y = x + y \mod \mathbb{Z}^N \quad \forall (x,y) \in \mathbb{R}^N \times \Omega.$$

Then it is easily checked that τ_x preserves the measure P for all $x \in \mathbb{R}^N$, and that the group transformation is ergodic. Thus the periodic setting is a particular case of the stationary ergodic setting.

We will denote by $E[\cdot]$ the expectation with respect to the probability measure P; in the periodic case, we will write $\langle f \rangle$ rather than $E[f]$ to refer to the average of f over one period.

We consider a potential function $u = u(y,\omega) \in L^\infty(\mathbb{R}^N \times \Omega)$ which is assumed to be stationary, i.e.

$$u(y+z,\omega) = u(y, \tau_z\omega) \quad \forall (y,z,\omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega.$$

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Moreover, we assume that $0 \le u(y,\omega) \le u_{\max} = \sup u\ \forall y \in \mathbb{R}^N, \omega \in \Omega$, and $u(\cdot) \in W^{2,\infty}_{\rm loc}(\mathbb{R}^N, L^\infty(\Omega))$, so that there exists a set $A \subset \Omega$ such that $P(A) = 0$, and $\nabla_y u(\cdot,\omega)$ is well-defined and locally Lipschitz continuous on $\mathbb{R}^N$, uniformly for $\omega \in \Omega \setminus A$.

Let $f^\varepsilon = f^\varepsilon(t,x,\xi,\omega)$, $(t \ge 0,\ x \in \mathbb{R}^N, \xi \in \mathbb{R}^N, \omega \in \Omega)$ be the solution of the transport equation

$$\begin{cases} \partial_t f^\varepsilon(t,x,\xi,\omega) + \xi \cdot \nabla_x f^\varepsilon(t,x,\xi,\omega) - \dfrac{1}{\varepsilon}\nabla_y u\left(\dfrac{x}{\varepsilon},\omega\right) \cdot \nabla_\xi f^\varepsilon(t,x,\xi,\omega) = 0, \\ f^\varepsilon(t=0,x,\xi,\omega) = f_0\left(x, \dfrac{x}{\varepsilon}, \xi, \omega\right). \end{cases} \tag{1}$$

Here, we assume that the initial data $f_0 = f_0(x,y,\xi,\omega)$ belongs to $L^1_{\rm loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$ and is stationary in y, i.e.

$$f_0(x, y+z, \xi, \omega) = f_0(x,y,\xi,\tau_z\omega) \quad \text{for all } (x,y,z,\xi,\omega) \in \mathbb{R}^{4N} \times \Omega.$$

We set $F_0(x,\xi,\omega) = f_0(x,0,\xi,\omega)$, and we also assume that f_0 is such that for all $\varepsilon > 0$, the function

$$f^\varepsilon(t=0) : (x,\xi,\omega) \mapsto f_0\left(x, \frac{x}{\varepsilon}, \xi, \omega\right)$$

belongs to $L^1_{\rm loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^1(\Omega))$; this fact does not follow directly from the above assumptions because in general, the measurability of $f^\varepsilon(t=0)$ is not clear. However, if $F_0 \in \mathcal{C}_c(\mathbb{R}^N_x, L^1_{\rm loc}(\mathbb{R}^N_\xi, L^\infty(\Omega)))$, for instance, then $f^\varepsilon(t=0)$ is measurable and belongs to $L^1_{\rm loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^1(\Omega))$. We will not comment further on this restriction and we refer to [4] for other sufficient assumptions on F_0. When f_0 satisfies the properties listed above, we say that f_0 is an *admissible initial data*; it can be checked (see [4]) that any linear combination of functions of the type

$$\chi_1(x)\, \chi_2(y,\xi,\omega)$$

with $\chi_1 \in L^1_{\rm loc}(\mathbb{R}^N)$ and $\chi_2 \in L^1_{\rm loc}(\mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$, with χ_2 stationary, is an admissible initial data.

It is well-known from the classical theory of linear transport equations that for every $\omega \in \Omega$, there exists a unique solution f^ε of (1) in $L^\infty_{\rm loc}((0,\infty), L^1_{\rm loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi))$. The goal of this paper is to study the asymptotic behavior of f^ε as $\varepsilon \to 0$. Thus, following [10], we define the constraint space $\mathbb{K}$:

Definition 1.1. *Let*

$$\xi \cdot \nabla_y f(y,\xi,\omega) - \nabla_y u(y,\omega) \cdot \nabla_\xi f(y,\xi,\omega) = 0 \tag{2}$$

be the constraint equation, and let

$$\mathbb{K} := \{ f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega));\ f \text{ satisfies (2) in } \mathcal{D}'(\mathbb{R}^N_y \times \mathbb{R}^N_\xi) \text{ a.s. in } \omega \}.$$

We also define the projection $\mathbb{P}$ onto the constraint space $\mathbb{K}$, characterised by $\mathbb{P}(f) \in \mathbb{K}$ for $f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega))$ stationary, and

$$\int_{\mathbb{R}^N \times \Omega} (\mathbb{P}(f) - f)(y,\xi,\omega)\, g(y,\xi,\omega)\, d\xi\, dP(\omega) = 0 \quad \text{for a.e. } y \in \mathbb{R}^N$$

for all stationary functions $g \in L^\infty(\mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega) \cap \mathbb{K}$, with compact support in ξ.

(A more precise definition of the projection $\mathbb{P}$ will be given in the second section).

Finally, we define $\mathbb{K}^\perp$ as

$$\mathbb{K}^\perp := \{ f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega)); \exists g \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega)), \quad f = \mathbb{P}(g) - g \}.$$

Remark 1.1. *Let us indicate that the constraint equation can easily be derived thanks to a formal two-scale Ansatz : indeed, assume that*

$$f^\varepsilon(t,x,\xi,\omega) \approx f\left(t, x, \frac{x}{\varepsilon}, \xi, \omega\right) \quad \text{as } \varepsilon \to 0;$$

inserting this asymptotic expansion in equation (1), we see that f necessarily satisfies the constraint equation (2).

Remark 1.2. *Let $f, g \in L^\infty(\mathbb{R}^N_y, L^2(\mathbb{R}^N_\xi \times \Omega))$ be stationary, and assume that $f \in \mathbb{K}$ and $g \in \mathbb{K}^\perp$. Then for a.e. $y \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N \times \Omega} f(y,\xi,\omega) g(y,\xi,\omega)\, d\xi\, dP(\omega) = 0.$$

This is a characterization of $\mathbb{K}^\perp$ for the class of stationary functions in $L^\infty(\mathbb{R}^N_y, L^2(\mathbb{R}^N_\xi \times \Omega))$.

Here, we provide another proof for the result of E. Frénod and K. Hamdache in [10] in the "non-perturbed case". Our proof is based on the use of the ergodic theorem, and gives a more concrete insight of the projection $\mathbb{P}$ and of the microscopic behavior of the sequence f^ε. Moreover, it allows us to retrieve the explicit formulas of the integrable case.

The first result we prove in this paper is the following theorem :

Theorem 1. *Let $f_0 \in L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y; L^1(\Omega))$ stationary, such that f_0 is an admissible initial data.*

Let $f^\varepsilon = f^\varepsilon(t,x,\xi,\omega)$ be the solution of (1). Then there exist two functions $f = f(t,x,y,\xi,\omega)$ and $g = g(t,x;\tau,y,\xi,\omega)$, both stationary in y, and a sequence $\{r^\varepsilon(t,x,\xi,\omega)\}_{\varepsilon>0}$ such that for all $\varepsilon > 0$

$$f^\varepsilon(t,x,\xi,\omega) = f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) + g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) + r^\varepsilon(t,x,\xi,\omega)$$

and :

- $||r^\varepsilon||_{L^1_{loc}((0,\infty)\times\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^1(\Omega))} \to 0$ *as* $\varepsilon \to 0$;
- $f \in L^\infty_{loc}((0,\infty); L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y; L^1(\Omega)))$, *and* $f(t,x) \in \mathbb{K}$ *for a.e.* $t \geq 0$, $x \in \mathbb{R}^N$;
- *For all $T > 0$, for all compact set $K \subset \mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y$,*

  $$\sup_{0\leq t\leq T, 0\leq\tau\leq T} ||g||_{L^1(K\times\Omega)} < \infty.$$

  Moreover, $g(t,x;\tau,\cdot) \in \mathbb{K}^\perp$ for a.e. $(t,x,\tau) \in (0,\infty) \times \mathbb{R}^N \times (0,\infty)$;
- *Microscopic evolution equation for g : for a.e. $t, x \in (0,\infty) \times \mathbb{R}^N$, $g(t,x;\cdot)$ is a solution of*

  $$\frac{\partial g}{\partial \tau} + \xi \cdot \nabla_y g - \nabla_y u \cdot \nabla_\xi g = 0. \tag{3}$$

  Moreover, for all $T > 0$

  $$\left|\left| \int_0^T g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) dt \right|\right|_{L^1_{loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^1(\Omega))} \to 0 \quad \text{as } \varepsilon \to 0.$$
- *Macroscopic evolution equation : f and g satisfy*

  $$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} + \xi^\sharp(y,\xi,\omega) \cdot \nabla_x \begin{pmatrix} f \\ g \end{pmatrix} = 0, \tag{4}$$

  where

  $$\xi^\sharp(y,\xi,\omega) := \mathbb{P}(\xi)(y,\xi,\omega);$$
- *Initial data :*

  $$f(t=0,x,y,\xi,\omega) = \mathbb{P}(f_0)(x,y,\xi,\omega),$$
  $$g(t=0,x;\tau=0,y,\xi,\omega) = \left[f_0 - \mathbb{P}(f_0)\right](x,y,\xi,\omega).$$

Before going any further, we wish to make a few comments on the above results. First, let us stress that it is not obvious that the function g is well-defined : indeed, let $S(t)$ ($t \geq 0$) denote the semi-group associated to the macroscopic evolution equation (4), and let $T(\tau)$ ($\tau \geq 0$) be the semi-group associated to the microscopic evolution equation (3). Then g is well defined if and only if, for all stationary function $g_0 = g_0(x, y, \xi, \omega)$, for all $t, \tau \geq 0$,

$$T(\tau)\left[S(t)g_0\right] = S(t)\left[T(\tau)g_0\right].$$

This identity follows from the fact that the speed $\xi^\sharp(y, \xi, \omega)$ appearing in equation (4) is a stationary solution of (3) by definition of the projection $\mathbb{P}$, and is thus invariant by the semi-group $T(\tau)$.

Next, let us explain briefly the meaning of theorem 1. The idea is the following : write f_0 as $f_0 = f_{0\|} + f_{0\perp}$, with $f_{0\|}(x, \cdot) \in \mathbb{K}$ and $f_{0\perp}(x, \cdot) \in \mathbb{K}^\perp$ a.e. Assume that $f_{0\|}$ and $f_{0\perp}$ are admissible initial data. Then f^ε can be written as $f^\varepsilon_\| + f^\varepsilon_\perp$, where $f^\varepsilon_\|$ (resp. $f^\varepsilon_\perp$) is the solution of equation (1) with initial data $f_{0\|}\left(x, x/\varepsilon, \xi, \omega\right)$ (resp. $f_{0\perp}\left(x, x/\varepsilon, \xi, \omega\right)$). Theorem 1 states that

$$f^\varepsilon_\| - f\left(t, x, \frac{x}{\varepsilon}, \xi, \omega\right) \to 0$$

strongly in L^1_{loc} norm. In particular, there are no microscopic oscillations in time in this part of f^ε. We wish to emphasize that this result appears to us to be new.

We now focus on the other part, namely $f^\varepsilon_\perp$. An easy consequence of the theorem is

$$\int_0^T f^\varepsilon_\perp(t, x, \xi, \omega)\, dt \to 0$$

in $L^\infty_{\text{loc}}(\mathbb{R}^N_x; L^1_{\text{loc}}(\mathbb{R}^N_\xi; L^1(\Omega)))$ and for all $T > 0$. However, it would be wrong to think that $f^\varepsilon_\perp$ vanishes in $L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^1(\Omega))$, for instance. Indeed

$$f^\varepsilon_\perp(t, x, \xi, \omega) \approx g\left(t, x; \frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)$$

in L^1_{loc}, and

$$||g(t = 0, x; \tau, y)||_{L^1(\mathbb{R}^N_\xi \times \Omega)} = ||f_{0\perp}(x, y)||_{L^1(\mathbb{R}^N_\xi \times \Omega)}$$

as soon as $f_{0\perp}(x, y) \in L^1(\mathbb{R}^N_\xi \times \Omega)$ for almost every x, y. And if $f_{0\perp}$ is stationary, then

$$||f_{0\perp}(x, y)||_{L^1(\mathbb{R}^N_\xi \times \Omega)} = ||f_{0\perp}(x, 0)||_{L^1(\mathbb{R}^N_\xi \times \Omega)} \quad \text{for a.e. } x, y.$$

Thus, if $f_{0\perp} \neq 0$, then for all $T > 0$, we derive

$$\begin{aligned}
\int_0^T \left|\left|g\left(t, x; \frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)\right|\right|_{L^1(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \Omega)} dt &= \int_0^T \left|\left|g\left(t, x; \frac{t}{\varepsilon}, 0, \xi, \omega\right)\right|\right|_{L^1(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \Omega)} dt \\
&= \int_0^T \left|\left|g\left(t = 0, x; \frac{t}{\varepsilon}, 0, \xi, \omega\right)\right|\right|_{L^1(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \Omega)} dt \\
&= T||f_{0\perp}(x, 0)||_{L^1(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \Omega)}.
\end{aligned}$$

The same kind of inequality holds if the L^1 norms are replaced with L^1_{loc} norms, but the derivation of the inequality with L^1_{loc} norms involves bounds on the function $\xi^\sharp$ which will be given later. Thus we refer to the proof of lemma 3.3 for arguments which yield similar inequalities with L^1_{loc} norms.

Hence $f^\varepsilon_\perp$ does not vanish strongly in general. In other words, there are fast oscillations in time, due to the ill-preparedness of the initial data (i.e. the fact that $f_0(x, \cdot) \notin \mathbb{K}$), but these oscillations do not cancel out as the small parameter ε vanishes.

Remark 1.3. *Some of the equations of theorem 1 can be guessed thanks to a two-scale Ansatz. We have already explained how equation* (2) *is obtained. The derivation of the evolution equation* (3) *is the same* *as the one of* (2)*, except that the Ansatz now involves microscopic oscillations in time. In other words, if*

$$f^\varepsilon(t,x,\xi,\omega) = f^0\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) + \varepsilon f^1\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) + \cdots,$$

then $f^0(t,x,\cdot)$ *satisfies* (3)*.*

The derivation of (4) *is less obvious, and in fact we have only been able to compute it for the function* f*, that is, when there are no microscopic oscillations in the time variable. Hence, assume that*

$$f^\varepsilon(t,x,\xi,\omega) = f^0\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) + \varepsilon f^1\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) + \cdots.$$

We insert this expansion in equation (1)*, and compute the* ε^0 *order term. We obtain*

$$\partial_t f^0 + \xi\cdot\nabla_x f^0 + \xi\cdot\nabla_y f^1 - \nabla_y u\cdot\nabla_\xi f^1 = 0$$

Thanks to remark 1.2, the term $\varphi^1(y,\xi,\omega) := \xi\cdot\nabla_y f^1 - \nabla_y u\cdot\nabla_\xi f^1$ *belongs to* $\mathbb{K}^\perp$*. Indeed, let* $g\in\mathbb{K}$ *be stationary and smooth, with compact support in* ξ*. According to Birkhoff's ergodic theorem, we have, for all* $y\in\mathbb{R}^N$*, and almost surely in* ω

$$\int_{\mathbb{R}^N\times\Omega}\varphi^1(y,\xi,\omega)g(y,\xi,\omega)\,d\xi\,dP(\omega) = \lim_{R\to\infty}\frac{1}{|B_R|}\int_{B_R\times\mathbb{R}^N_\xi}\varphi^1(z,\xi,\omega)g(z,\xi,\omega)\,dz\,d\xi.$$

And if f^1 *and* g *are smooth,*

$$\begin{aligned}\int_{B_R\times\mathbb{R}^N_\xi}\varphi^1(z,\xi,\omega)g(z,\xi,\omega)\,dz\,d\xi &= -\int_{B_R\times\mathbb{R}^N_\xi}\left(\xi\cdot\nabla_z g - \nabla_z u\cdot\nabla_\xi g\right)(z,\xi,\omega)f^1(z,\xi,\omega)\,dz\,d\xi\\ &\quad+\int_{\partial B_R}\int_{\mathbb{R}^N}\xi\cdot n_{B_R}(y)g(z,\xi,\omega)f^1(z,\xi,\omega)\,dz\,d\xi.\end{aligned}$$

Since $g\in\mathbb{K}$ *and* g *has compact support in* ξ*, we deduce that* $\varphi^1\in\mathbb{K}^\perp$ *thanks to remark 1.2. Recall also that* $f^0(t,x,\cdot)$ *belongs to* $\mathbb{K}$ *almost everywhere. Thus projecting the above equation on* $\mathbb{K}$ *yields* (4)*.*

Let us now explain how our strategy of proof differs from the one of E. Frénod and K. Hamdache in [10]. The authors of [10] used the concept of two-scale convergence, a notion introduced by Gabriel N'Guetseng in [16], and then formalized and developed by Grégoire Allaire in [1]. We will first explain briefly what are the main arguments of [10], and then we shall expose the great lines of the proof of the present paper.

The notion of two-scale convergence relies on a choice of oscillating test functions; the central result of the theory is the following (see [1]) :

Proposition 1.1. *Let* U *be an open set in* $\mathbb{R}^N$*, and let* $(g^\varepsilon)_{\varepsilon>0}$ *be a bounded sequence in* $L^2(U)$*. Then there exists a function* $g^0\in L^2(U\times[0,1]^N)$*, and a subsequence* (ε_n) *such that* $\varepsilon_n\to0$ *as* $n\to\infty$*, such that*

$$\int_U g^{\varepsilon_n}(x)\psi\left(x,\frac{x}{\varepsilon_n}\right)\,dx \to \int_{U\times[0,1]^N} g^0(x,y)\psi(x,y)\,dx\,dy$$

for all functions $\psi\in L^2(U;\mathcal{C}_{per}([0,1]^N))$*.*

It is then said that the sequence g^{ε_n} *two-scale converges towards* g_0*.*

This concept can be generalized with no difficulty to functions depending on t and ξ as well. In [10], the authors pass to the two-scale limit in equation (1), after deriving *a priori* bounds on the sequence f^ε. Using test functions of the type

$$\varepsilon\varphi\left(t,x,\frac{x}{\varepsilon},\xi\right),$$

with $\varphi\in\mathcal{D}_{\rm per}([0,\infty)\times\mathbb{R}^N_x\times[0,1]^N\times\mathbb{R}^N_\xi)$, they first prove that the two-scale limit of the sequence f^ε, say $f(t,x,y,\xi)$, satisfies the constraint equation (2). Then, taking test functions of the type

$$\varphi\left(t,x,\frac{x}{\varepsilon},\xi\right)$$

such that $\varphi(t,x,\cdot) \in \mathbb{K}$ almost everywhere, they derive the macroscopic evolution equation (4). The proof is straight-forward and simpler than the one we present here, but does not include any description of the microscopic oscillations in time. Moreover, since the method of [10] relies on two-scale convergence, the result only provides information on the weak-limit (or two-scale limit) of the sequence f^ε; in other words, the strong convergence result we state here is new, and cannot be derived by two-scale convergence techniques.

However, let us mention that the notion of two-scale convergence has been generalized to stationary settings by Alain Bourgeat, Andro Mikelić and Steve Wright (see [4]); the relevant concept is then called *stochastic two-scale convergence in the mean.* The result of [4] is the following :

Proposition 1.2. *Assume that $L^2(\Omega, P)$ is separable.*

Let U be an open set in $\mathbb{R}^N$, and let $(g^\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^2(U\times\Omega)$. Then there exists a function $g^0 \in L^2(U\times\Omega)$, and a subsequence (ε_n) such that $\varepsilon_n \to 0$ as $n\to\infty$, such that

$$\int_{U\times\Omega} g^{\varepsilon_n}(x,\omega)\psi\left(x, \tau_{\frac{x}{\varepsilon_n}}\omega\right)\, dx\, dP(\omega) \to \int_{U\times\Omega} g^0(x,\omega)\psi(x,\omega)\, dx\, dP(\omega),$$

for any $\psi \in L^2(U\times\Omega)$ such that the function

$$(x,\omega) \mapsto \psi(x, \tau_x\omega)$$

belongs to $L^2(U\times\Omega)$.

It is likely that the arguments of [10] can be generalized to the present case in order to obtain the same kind of weak convergence results, as long as $L^2(\Omega)$ is separable. We prefer to focus on a different method, which is more explicit and which allows for the derivation of strong convergence results.

The key of our analysis lies in the study of the behavior as $\varepsilon\to 0$ of the Hamiltonian system

$$\begin{cases} \dot{Y}^\varepsilon(t,x,\xi,\omega) = -\Xi^\varepsilon(t,x,\xi,\omega), \quad t>0 \\ \dot{\Xi}^\varepsilon(t,y,\xi,\omega) = \frac{1}{\varepsilon}\nabla_y u(Y^\varepsilon(t,x,\xi,\omega),\omega), \quad t>0 \\ Y^\varepsilon(t=0,x,\xi,\omega) = x,\ \Xi^\varepsilon(t=0,x,\xi,\omega) = \xi, \quad (x,\xi,\omega)\in\mathbb{R}^{2N}\times\Omega. \end{cases}$$

Indeed, if f_0 is smooth, then

$$f^\varepsilon(t,x,\xi,\omega) = f_0\left(Y^\varepsilon(t,x,\xi,\omega), \frac{Y^\varepsilon(t,x,\xi,\omega)}{\varepsilon}, \Xi^\varepsilon(t,y,\xi,\omega), \omega\right),$$

so that we can deduce the asymptotic behavior of f^ε from the one of $(Y^\varepsilon, \Xi^\varepsilon)$. And it is easily checked that

$$Y^\varepsilon(t,x,\xi,\omega) = \varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right),$$
$$\Xi^\varepsilon(t,y,\xi,\omega) = \Xi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right),$$

where (Y,Ξ) is the solution of the system

$$\begin{cases} \dot{Y}(t,y,\xi,\omega) = -\Xi(t,y,\xi,\omega), \quad t>0, \\ \dot{\Xi}(t,y,\xi,\omega) = \nabla_y u(Y(t,y,\xi,\omega),\omega), \quad t>0, \\ Y(t=0,y,\xi,\omega) = y,\ \Xi(t=0,y,\xi,\omega) = \xi, \quad (y,\xi,\omega)\in\mathbb{R}^{2N}\times\Omega. \end{cases} \tag{5}$$

Hence, in order to study the limit of f^ε as $\varepsilon\to 0$, we have to investigate the long time behavior of the system (Y,Ξ), and this will be achieved with the help of the ergodic theorem in the second section.

This dynamical system also allows for a better understanding of the function $\xi^\sharp$ appearing in (4). Indeed, we shall prove in the second section that

$$\xi^\sharp(y,\xi,\omega) = -\lim_{T\to\infty} \frac{Y(T,y,\xi,\omega)}{T} \quad \text{almost everywhere,}$$

so that $-\xi^\sharp$ is the rotation vector associated with the dynamics (Y,Ξ).

In the case where $N = 1$, we can give explicit formulas for $\xi^\sharp(y,\xi,\omega)$; the proof of this formula in the stationary ergodic case is strongly linked to methods from the Aubry-Mather theory (see [8], [9], [15]), and thus also to the homogenization of Hamilton-Jacobi equations. In the rest of the paper, we set

$$H(y,\xi,\omega) = \frac{|\xi|^2}{2} + u(y,\omega).$$

Let us first recall the definition of the homogenized Hamiltonian $\bar{H}$ (see [13])

$$\bar{H}(p) = u_{\max} + \frac{1}{2}\begin{cases} 0 & \text{if } |p| < E\left[\sqrt{2(u_{\max}-u)}\right] \\ \lambda & \text{if } |p| \geq E\left[\sqrt{2(u_{\max}-u)}\right], \quad \text{where } |p| = E\left[\sqrt{2(u_{\max}-u)+\lambda}\right]. \end{cases}$$

Proposition 1.3. *Assume that $N = 1$.*

Let $(y,\xi,\omega) \in \mathbb{R}\times\mathbb{R}\times\Omega$ such that $H(y,\xi,\omega) > u_{max}$. Let $P = P(y,\xi,\omega) \in \mathbb{R}$ such that $\bar{H}(P) = H(y,\xi,\omega)$ and $\operatorname{sgn}(P) = \operatorname{sgn}(\xi)$. Then

$$\xi^\sharp(y,\xi,\omega) = \bar{H}'(P)$$

Moreover, if L is the dual function of H, i.e.

$$L(y,p,\omega) = \sup_{\xi\in\mathbb{R}} (p\xi - H(y,\xi\,\omega)) = \frac{1}{2}|p|^2 - u(y,\omega),$$

and $\bar{L}$ is the homogenized Lagrangian (the dual function of $\bar{H}$), then

$$\mathbb{P}(L)(y,\xi,\omega) = \bar{L}(\xi^\sharp(y,\xi,\omega).)$$

In the periodic case, we will give another proof of the above result; the strategy chosen in that case is inspired from techniques and calculations in classical mechanics. It also allows to give a formula for $\xi^\sharp$ for low energies in the periodic setting only:

Proposition 1.4. *Assume that $N = 1$ and that the environment is periodic.*

Let $(y,\xi) \in \mathbb{R}^2$ such that $H(y,\xi) < u_{max}$. Then $\xi^\sharp(y,\xi) = 0$.

The organisation of this note is the following : in the second section, we derive some preliminary results on the long-time behavior of the system (Y,Ξ) thanks to the ergodic theorem. Those will be useful in the proof of theorem 1, to which is devoted the third section. Eventually, the fourth and last section is concerned with results in the integrable case, both in the periodic and the stationary ergodic settings.

2 Preliminaries

This section is largely devoted to the study of the long-time behavior of the Hamiltonian system (Y,Ξ) defined by (5). First, notice that the Hamiltonian $H(y,\xi,\omega) := \frac{1}{2}|\xi|^2 + u(y,\omega)$ is constant along the curves of the system (Y,Ξ), and if $f \in L^\infty(\Omega, \mathcal{C}^1(\mathbb{R}^N_y \times \mathbb{R}^N_\xi))$ is stationary, then

$$f \in \mathbb{K} \iff f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega),\omega\right) = f(y,\xi,\omega) \quad \forall (y,\xi,\omega) \in \mathbb{R}^N\times\mathbb{R}^N\times\Omega.$$

Indeed, for all $f \in L^\infty(\Omega, \mathcal{C}^1(\mathbb{R}^N_y \times \mathbb{R}^N_\xi))$, we have

$$\frac{\partial}{\partial t} f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega),\omega\right) = \{H,f\}\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega),\omega\right),$$

where $\{H,f\}$ denotes the Poisson bracket of f and H, i.e.

$$\{H,f\}(y,\xi,\omega) = \xi\cdot\nabla_y f(y,\xi,\omega) - \nabla_y u(y,\xi,\omega)\cdot\nabla_\xi f(y,\xi,\omega).$$

Let us mention an easily checked property of the trajectories (Y, Ξ) which will be used extensively in the rest of the article : for all $(y, z, \xi) \in \mathbb{R}^{3N}$, for all $\omega \in \Omega$, $t \geq 0$,

$$\begin{aligned} Y(t, y, \xi, \tau_z \omega) + z &= Y(t, y + z, \xi, \omega), \\ \Xi(t, y, \xi, \tau_z \omega) &= \Xi(t, y + z, \xi, \omega). \end{aligned} \tag{6}$$

In the periodic case, this invariance entails that the hamiltonian system (Y, Ξ) can be considered as a dynamical system on the N dimensional torus $[0, 2\pi)^N$. In this periodic setting, it is somewhat natural to introduce the semi-group of transformations $(\mathcal{T}_t)_{t\geq 0}$ on $[0, 2\pi)^N \times \mathbb{R}^N$ given by

$$\mathcal{T}_t(y, \xi) = (Y(t, y, \xi), \Xi(t, y, \xi)), \quad y \in [0, 2\pi)^N, \ \xi \in \mathbb{R}^N.$$

According to Liouville's theorem, this semi-group preserves the Lebesgue measure on $[0, 2\pi)^N \times \mathbb{R}^N$; moreover, we can construct a family of finite invariant measures on $[0, 2\pi)^N \times \mathbb{R}^N$ by setting $m_c(y, \xi) = \mathbf{1}_{H(y,\xi)\leq c}\, dy\, d\xi$ for $c > 0$ (remember that the Hamiltonian is constant along the hamiltonian curves). This construction is the root of the ergodic theorem (see corollary 2.1), and thus of the study of the long-time behavior of the system (Y, Ξ).

In the stationary ergodic setting, this construction can be generalized as follows : we define the transformation $T_t : \mathbb{R}^N_\xi \times \Omega \to \mathbb{R}^N_\xi \times \Omega$ by

$$T_t(\xi, \omega) = \left(\Xi(t, 0, \xi, \omega), \tau_{Y(t,0,\xi,\omega)}\omega\right)$$

together with the family of measures

$$\mu_c := \mathbf{1}_{\mathcal{H}(\xi,\omega)\leq c}\, d\xi\, dP(\omega)$$

where $\mathcal{H}(\xi, \omega) := \frac{1}{2}|\xi|^2 + u(0, \omega)$. It is obvious that for all $c \in (0, \infty)$, μ_c is a finite measure on $\mathbb{R}^N_\xi \times \Omega$.

Notice that the "good" generalization to the stationary ergodic setting of the semi-group $(\mathcal{T}_t)$ is a semi-group which acts on $\mathbb{R}^N_\xi \times \Omega$ rather than $\mathbb{R}^N_y \times \mathbb{R}^N_\xi$. Thanks to the group of transformations $(\tau_x)_{x\in\mathbb{R}^N}$, the transformations in Ω can result in transformations in $\mathbb{R}^N_y$, but the definition chosen here allows us to define a family of finite invariant measures, whereas such a construction is rather difficult if one tries to define a semi-group acting on $\mathbb{R}^N_y \times \mathbb{R}^N_\xi$. This will be fundamental in the rest of the proof.

Lemma 2.1. *$(T_t)_{t\geq 0}$ is a semi-group on $\mathbb{R}^N_\xi \times \Omega$ and preserves the family of measures μ_c.*

Proof. Let us first prove the semi-group property : let $t, s \in [0, \infty)$, and $(\xi, \omega) \in \mathbb{R}^N \times \Omega$; then

$$\begin{aligned} T_t \circ T_s(\xi, \omega) &= T_t\left(\Xi(s, 0, \xi, \omega), \tau_{Y(s,0,\xi,\omega)}\omega\right) \\ &= \left(\Xi(t, 0, \Xi(s, 0, \xi, \omega), \tau_{Y(s,0,\xi,\omega)}\omega), \omega'\right) \end{aligned}$$

and using the properties (6) we deduce

$$\begin{aligned} \Xi(t, 0, \Xi(s, 0, \xi, \omega), \tau_{Y(s,0,\xi,\omega)}\omega) &= \Xi(t, Y(s, 0, \xi, \omega), \Xi(s, 0, \xi, \omega), \omega), \\ &= \Xi(t + s, 0, \xi, \omega) \end{aligned}$$

and

$$\begin{aligned} \omega' &= \tau_{Y(t,0,\Xi(s,0,\xi,\omega),\tau_{Y(s,0,\xi,\omega)}\omega)}\tau_{Y(s,0,\xi,\omega)}\omega \\ &= \tau_{Y(t,0,\Xi(s,0,\xi,\omega),\tau_{Y(s,0,\xi,\omega)}\omega)+Y(s,0,\xi,\omega)}\omega \\ &= \tau_{Y(t,Y(s,0,\xi,\omega),\Xi(s,0,\xi,\omega),\omega)}\omega \\ &= \tau_{Y(t+s,0,\xi,\omega)}\omega \end{aligned}$$

Thus

$$T_t \circ T_s(\xi, \omega) = \left(\Xi(t + s, 0, \xi, \omega), \tau_{Y(t+s,0,\xi,\omega)}\omega\right) = T_{t+s}(\xi, \omega).$$

Since it is obvious that $T_0 = \mathrm{Id}$, $(T_t)_{t\geq 0}$ is a semi-group on $\mathbb{R}^N \times \Omega$.

We now have to check the invariance property; let $F \in L^1(\mathbb{R}^N \times \Omega; \mu_c)$ arbitrary. We set $f(y,\xi,\omega) := F(\xi, \tau_y\omega)$ for $(y,\xi,\omega) \in \mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega$, and we compute

$$\int_{\mathbb{R}^N\times\Omega} F(T_t(\xi,\omega))\, d\mu_c(\xi,\omega) = E\left[\int_{R^N} f(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\mathbf{1}_{H(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\le c}\, d\xi\right].$$

Since the probability measure P is invariant by the group of transformation τ_y, and

$$f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega) = f(Y(t,0,\xi,\tau_y\omega),\Xi(t,0,\xi,\tau_y\omega),\tau_y\omega),$$

we have, for all $y \in \mathbb{R}^N$

$$\begin{aligned} &E\left[f(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\mathbf{1}_{H(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\le c}\right] = \\ &\qquad = E\left[f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\mathbf{1}_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\le c}\right].\end{aligned}$$

Take an arbitrary function $\phi \in L^1(\mathbb{R}^N_y)$, and write

$$\begin{aligned} &\int_{\mathbb{R}^N\times\Omega} F(T_t(\xi,\omega))\, d\mu_c(\xi,\omega) \\ = \;& E\left[\int_{\mathbb{R}^{2N}} dy\, d\xi \phi(y) f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\mathbf{1}_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\le c}\right]\end{aligned}$$

We change variables in the integral in (y,ξ) by setting $(x,v) = (Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega))$; according to Liouville's theorem, the jacobian of this change of variables is equal to 1, and

$$(x,v) = (Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega)) \iff (y,\xi) = (X(t,x,v,\omega),V(t,x,v,\omega)),$$

where (X,V) is a solution of the Hamiltonian system

$$\begin{cases} \dot{X} = V, \\ \dot{V} = -\nabla u(X,\omega), \\ (X,V)(t=0,x,v) = (x,v). \end{cases}$$

Observe that in the present case, we have simply

$$X(t,x,v,\omega) = Y(t,x,-v,\omega),$$

so that (X,V) satisfies relations (6).

Hence

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} dy\, d\xi\; \phi(y) f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\mathbf{1}_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\le c} \\ = &\int_{\mathbb{R}^{2N}} dx\, dv\; \phi(X(t,x,v,\omega)) f(x,v,\omega)\mathbf{1}_{H(x,v,\omega)\le c} \\ = &\int_{\mathbb{R}^{2N}} dx\, dv\; \phi(X(t,0,v,\tau_x\omega)+x) F(v,\tau_x\omega)\mathbf{1}_{\mathcal{H}(v,\tau_x\omega)\le c}\end{aligned}$$

so that

$$\begin{aligned} &\int_{\mathbb{R}^N\times\Omega} F(T_t(\xi,\omega))\, d\mu_c(\xi,\omega) \\ = \;& E\left[\int_{\mathbb{R}^{2N}} dx\, dv\; \phi(X(t,0,v,\tau_x\omega)+x) F(v,\tau_x\omega)\mathbf{1}_{\mathcal{H}(v,\tau_x\omega)\le c}\right] \\ = \;& E\left[\int_{\mathbb{R}^{2N}} dx\, dv\; \phi(X(t,0,v,\omega)+x) F(v,\omega)\mathbf{1}_{\mathcal{H}(v,\omega)\le c}\right] \\ = \;& E\left[\int_{\mathbb{R}^N} dv\; \left(\int_{\mathbb{R}^N} \phi(X(t,0,v,\omega)+x)\, dx\right) F(v,\omega)\mathbf{1}_{\mathcal{H}(v,\omega)\le c}\right] \\ = \;& E\left[\int_{\mathbb{R}^N} dv F(v,\omega)\mathbf{1}_{\mathcal{H}(v,\omega)\le c}\right] = \int_{\mathbb{R}^N\times\Omega} F\, d\mu_c\end{aligned}$$

since the integral of ϕ is equal to 1.

Hence, we have proved that for all $F \in L^1(\mathbb{R} \times \Omega; \mu_c)$, for all $t \geq 0$,

$$\int_{\mathbb{R}^N \times \Omega} F(T_t(\xi, \omega))\, d\mu_c(\xi, \omega) = \int_{\mathbb{R}^N \times \Omega} F(\xi, \omega)\, d\mu_c(\xi, \omega),$$

which means exactly that μ_c is invariant by the semi-group $(T_t)_{t \geq 0}$.

□

The following corollary is an immediate consequence of Birkhoff's ergodic theorem:

Corollary 2.1. *Let $F \in L^1(\mathbb{R}^N \times \Omega; \mu_c)$. There exists a function $\bar{F} \in L^1(\mathbb{R}^N \times \Omega; \mu_c)$ such that as $T \to \infty$,*

$$\frac{1}{T} \int_0^T F(T_t(\xi, \omega))\, dt \to \bar{F}(\xi, \omega)$$

a.e. on $\mathbb{R}^N \times \Omega$ and in $L^1(\mu_c)$. Moreover, $\bar{F}$ is invariant by T_t for all $t > 0$, and

$$\int_{\mathbb{R}^N \times \Omega} F\, d\mu_c = \int_{\mathbb{R}^N \times \Omega} \bar{F}\, d\mu_c. \tag{7}$$

Additionally, if $\bar{f} = \bar{f}(y, \xi, \omega)$ is the stationary function associated to $\bar{F}$, that is, $\bar{f}(y, \xi, \omega) = \bar{F}(\xi, \tau_y \omega)$, then $\bar{f}$ is invariant by the hamiltonian flow (Y, Ξ); precisely, for a.e. $(y, \xi, \omega) \in \mathbb{R}^{2N} \times \Omega$, $t > 0$

$$\bar{f}(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega) = \bar{f}(y, \xi, \omega).$$

Proof. We only have to prove the invariance of $\bar{f}$ by the Hamiltonian flow; first, for $y = 0$, we have

$$\begin{aligned} \bar{f}(Y(t, 0, \xi, \omega), \Xi(t, 0, \xi, \omega), \omega) &= \bar{F}(\Xi(t, 0, \xi, \omega), \tau_{Y(t,0,\xi,\omega)}\omega) = \bar{F}(T_t(\xi, \omega)) \\ &= \bar{F}(\xi, \omega) = \bar{f}(0, \xi, \omega) \end{aligned}$$

and the property is proved when $y = 0$.

For $y \in \mathbb{R}^N$ arbitrary,

$$\begin{aligned} \bar{f}(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega) &= \bar{f}(Y(t, 0, \xi, \tau_y\omega) + y, \Xi(t, 0, \xi, \tau_y\omega), \omega) \\ &= \bar{f}(Y(t, 0, \xi, \tau_y\omega), \Xi(t, 0, \xi, \tau_y\omega), \tau_y\omega) \\ &= \bar{f}(0, \xi, \tau_y\omega) = \bar{f}(y, \xi, \omega) \end{aligned}$$

according to the result in the case $y = 0$.

□

Remark 2.1. *We mention here an important but easy consequence of the relations* (6) *and the invariance of the measure P with respect to the transformation group τ_y, $y \in \mathbb{R}^N$: for any stationary function $f = f(y, \xi, \omega) = F(\xi, \tau_y\omega)$, $F \in L^\infty(\mathbb{R}^N \times \Omega)$, we have*

$$E[f(Y(t, y, \xi, \cdot), \Xi(t, y, \xi, \cdot), \cdot)] = E[F(T_t(\xi, \cdot))]$$

for all $t > 0$, $y, \xi \in \mathbb{R}^N$; in particular, the left-hand side of the above equality does not depend on y.

This property was used in the proof of lemma 2.1

Remark 2.2. *Let us precise a little what happens when the function $F \in L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$. In that case, $F \in L^1(\mathbb{R}^N_\xi \times \Omega; \mu_c)$ for all $c > 0$. Consequently, for any $c > 0$, we can define the function $\bar{F}_c$ associated to F by corollary 2.1.*

It is then easily proved that for any $0 < c < c'$, $\bar{F}_c = \bar{F}_{c'}$, μ_c-almost everywhere. Setting

$$\mathrm{Supp}\mu_n := \{(\xi, \omega), \mathcal{H}(\xi, \omega) \leq n\},$$
$$A_n = \{(\xi, \omega) \in \mathrm{Supp}\mu_n;\ \bar{F}_n(y, \xi) \neq \bar{F}_{n+1}(y, \xi)\}$$
$$A := \bigcup_{n=0}^{\infty} A_n,$$

we see that $\mu_c(A) = 0$ *for all* $c > 0$*. Moreover, for all* $(\xi,\omega) \in \mathbb{R}^N \times \Omega \setminus A$*, for all integers* k, l *such that* $(\xi,\omega) \in \mathrm{Supp}\mu_k \cap \mathrm{Supp}\mu_l$*, we have* $\bar{F}_k(\xi,\omega) = \bar{F}_l(\xi,\omega)$*. We can thus define a function* $\bar{F}(\xi,\omega)$ *on* $\mathbb{R}^N \times \Omega \setminus A$ *by*

$$\bar{F}(\xi,\omega) = \bar{F}_n(\xi,\omega) \quad \textit{for any } n \in \mathbb{N} \textit{ such that } (\xi,\omega) \in \mathrm{Supp}\mu_n$$

We then now that

$$\frac{1}{T}\int_0^T F(T_t(\xi,\omega))\,dt \to \bar{F}(\xi,\omega) \tag{8}$$

as $T \to \infty$*, and the convergence holds in* $L^1(\mu_c)$ *for all* $c > 0$*, and* μ_n *almost everywhere for* $n \in \mathbb{N}$*. Eventually, setting*

$$B := \{(\xi,\omega) \in \mathbb{R}^N \times \Omega \setminus A;\ \frac{1}{T}\int_0^T F(T_t(\xi,\omega))\,dt \textit{ does not converge towards } \bar{F}(\xi,\omega) \textit{ as } T \to \infty\}$$

it is easily proved that $\mu_c(B) = 0$ *for all* $c > 0$ *(the equality is true for* $c \in \mathbb{N}$*, and is then deduced for* $c > 0$ *arbitrary because the family of measures* (μ_c) *is increasing in* c*).*

Eventually, we have found a function $\bar{F} \in L^1_{loc}(\mathbb{R}^N, L^1(\Omega))$*, independent of* c*, such that* (8) *holds in* $L^1(\mu_c)$ *and* μ_c*-almost everywhere for all* $c > 0$*.*

Remark 2.3. *The construction above allows us to make more precise what we mean by projection* $\mathbb{P}$*: let* $f = f(y,\xi,\omega)$ *be a stationary function,* $f \in L^\infty(\mathbb{R}^N_y, L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega)))$*, and set* $F(\xi,\omega) = f(0,\xi,\omega) \in L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$*. We can then associate to* F *a function* $\bar{F} \in L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$ *such that* (8) *holds in* $L^1(\mu_c)$ *for all* c *(see remark 2.2). We set*

$$\mathbb{P}(f)(y,\xi,\omega) := \bar{F}(\xi,\tau_y\omega).$$

It follows from corollary 2.1 that $\mathbb{P}(f)$ *is invariant by the hamiltonian flow* (5)*, and thus satisfies the constraint equation. From now on, we take this definition for the projection* $\mathbb{P}$*, instead of the one given in the introduction. Notice that, for all* $y \in \mathbb{R}^N$ *and* μ_c*-almost everywhere,*

$$\begin{aligned}
\mathbb{P}(f)(y,\xi,\omega) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T F(T_t(\xi,\tau_y\omega))\,dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(Y(t,0,\xi,\tau_y\omega), \Xi(t,0,\xi,\tau_y\omega), \tau_y\omega\right)\,dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega), \omega\right)\,dt
\end{aligned}$$

And we also give a more precise definition of $\xi^\sharp(y,\xi,\omega)$ *: let*

$$\hat{\xi}: \begin{array}{ccc} \mathbb{R}^N \times \Omega & \to & \mathbb{R}^N \\ (\xi,\omega) & \mapsto & \xi \end{array}.$$

Then $\hat{\xi} \in L^1(\mathbb{R}^N \times \Omega, \mu_c)$ *for all* $c > 0$*, and*

$$\xi^\sharp(y,\xi,\omega) = P(\hat{\xi})(y,\xi,\omega) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Xi(t,y,\xi,\omega)\,dt$$

almost everywhere and in $L^1(\mu_c)$ *for all* $0 < c < \infty$*.*

Eventually, we mention here a property that will be used in the proof of the theorem; with the same notations as above, let

$$\phi(\tau,y,\xi,\omega) = F\left(T_\tau(\xi,\tau_y\omega)\right),$$

with $F \in L^1_{\mathrm{loc}}(\mathbb{R}^N_\xi, L^1(\Omega))$. Then ϕ is a solution of the evolution equation

$$\partial_\tau\phi + \xi\cdot\nabla_y\phi - \nabla_y u\cdot\nabla_\xi\phi = 0,$$

with initial data $\phi(\tau = 0,y,\xi,\omega) = f(y,\xi,\omega) = F(\xi,\tau_y\omega)$. This is a classical fact if ϕ is $\mathcal{C}^1$ in the variables y,ξ; thanks to a contraction property for the transport equation (3) and a density result which will be stated later (see lemma 3.2), it is true when F merely belongs to L^1. In order to avoid technical details at this stage, we omit the proof of this result for weak (that is, L^1) solutions.

3 The general N-dimensional case

This section is devoted to the proof of theorem 1. The proof is divided in three steps : first, we study the case of an initial data which does not depend on x, then the case when the initial data only depends on x (and not on y,ξ,ω), and eventually, we treat the general case.

3.1 First case : f_0 does not depend on x

Here, we assume that $f_0=f_0(y,\xi,\omega)\in L^1_{\text{loc}}(\mathbb{R}^N_\xi;L^\infty(\mathbb{R}^N_y\times\Omega))\cap\mathcal{C}^1(\mathbb{R}^N_\xi\times\mathbb{R}^N_y,L^\infty(\Omega))$. The smoothness assumption will be removed in the third section. Recall that f_0 is stationary, that is, $f_0(y+z,\xi,\omega)=f_0(y,\xi,\tau_z\omega)$ a.s. in ω, for all $(y,z,\xi)\in\mathbb{R}^{3N}$. In the rest of the subsection, we set

$$F_0(\xi,\omega):=f_0(0,\xi,\omega)$$

and

$$\bar{F}_0(\xi,\omega):=\lim_{T\to\infty}\frac{1}{T}\int_0^T F_0(T_t(\xi,\omega))\,dt,\quad \bar{f}_0(y,\xi,\omega)=\bar{F}_0(\xi,\tau_y\omega).$$

Notice that $F_0\in L^1_{\text{loc}}(\mathbb{R}^N_\xi;L^\infty(\Omega))$, and thus $F_0\in L^1(\mathbb{R}^N\times\Omega;\mu_c)$ for all $c>0$.

In that case,

$$\begin{aligned}
f^\varepsilon(t,x,\xi,\omega) &= f_0\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\\
&= f_0\left(Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right),\Xi\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right),\tau_{\frac{x}{\varepsilon}}\omega\right)\\
&= F_0\left(T_{\frac{t}{\varepsilon}}\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right)\\
&= \bar{f}_0\left(\frac{x}{\varepsilon},\xi,\omega\right)+\left\{F_0\left(T_{\frac{t}{\varepsilon}}\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right)-\bar{F}_0\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right\}
\end{aligned}$$

In accordance with theorem 1, we set

$$g(\tau,y,\xi,\omega)=\left(F_0-\bar{F}_0\right)\left(T_\tau\left(\xi,\tau_y\omega\right)\right),$$

and $r^\varepsilon=0$. Then g satisfies the microscopic evolution equation (3) thanks to the remark at the end of the preceding section. Moreover, $g(\tau)\in\mathbb{K}^\perp$ by definition of $\mathbb{K}^\perp$ and because $\mathbb{P}\left(F_0\left(T_\tau\left(\xi,\tau_y\omega\right)\right)\right)=\bar{F}_0(\xi,\tau_y\omega)$. Notice also that $\bar{f}_0=\mathbb{P}(f_0)$ thanks to remark 2.3.

There only remains to check that

$$\int_0^T g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt\to 0\quad\text{as }\varepsilon\to 0 \tag{9}$$

in $L^1_{\text{loc}}(\mathbb{R}^N_x,L^1(\mathbb{R}^N\times\Omega,\mu_c))$ for all $T>0$ and $c>0$.

The invariance of the measure P with respect to the group of transformations $(\tau_x)_{x\in\mathbb{R}^N}$ (see remark 2.1) entails that

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^N_\xi}\left|\frac{1}{\frac{T}{\varepsilon}}\int_0^{\frac{T}{\varepsilon}} f_0\left(Y\left(t,\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(t,\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\,dt-\bar{f}_0\left(\frac{x}{\varepsilon},\xi,\omega\right)\right|\,d\mu_c(\xi,\omega)\\
=&\int_{\Omega\times\mathbb{R}^N_\xi}\left|\frac{1}{\frac{T}{\varepsilon}}\int_0^{\frac{T}{\varepsilon}} F_0\left(T_t(\xi,\omega)\right)\,dt-\bar{F}_0\left(\xi,\omega\right)\right|\,d\mu_c(\xi,\omega)
\end{aligned}$$

and the term above goes to 0 as $\varepsilon\to 0$ according to corollary 2.1 and is independent of $x\in\mathbb{R}^N$. Thus theorem 1 is proved in the case when f_0 does not depend on the macroscopic variable x.

The following remark will prove to be useful when treating the general case :

Remark 3.1. *If $f_0 \in L^\infty$ (and f_0 is $\mathcal{C}^1$ in the variables y, ξ), then for any function $a \in L^\infty((0,\infty) \times \mathbb{R}^N_x \times \mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega)$, stationary in y, we have*

$$\int_0^T a\left(t, x, \frac{x}{\varepsilon}, \xi, \omega\right) g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt \to 0 \quad \text{as } \varepsilon \to 0$$

in $L^1_{loc}(\mathbb{R}^N_x, L^1(\mathbb{R}^N_\xi \times \Omega, \mu_c))$ for all $T > 0$ and $c > 0$.

Indeed, let us first prove the property for $a = a_1(t) a_2(x, y, \xi, \omega)$, with a_1, $a_2 \in L^\infty$. If a_1 is an indicator function of the type

$$a_1(t) = \mathbf{1}_{T_1 < t < T_2},$$

the the property follows from the equality

$$\int_0^T a_1(t)\, g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt = \int_0^{\inf(T, T_2)} g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt - \int_0^{\inf(T, T_1)} g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt.$$

According to (9), the convergence result thus follows for $a(t, y, \xi, \omega) = a_1(t) a_2(y, \xi\omega)$. Then, let $a_1 \in L^\infty([0,\infty))$ be arbitrary, and let $T > 0$, $n \in \mathbb{N}^*$. Let $b_n \in L^\infty(([0,\infty))$ such that $||a_1 - b_n||_{L^1(0,T)} \leq 1/n$, and

$$b_n = \sum_{i=1}^{N_n} \alpha_{i,n} \mathbf{1}_{T_{i,n} < t < T_{i',n}},$$

with $N_n \in \mathbb{N}$, $\alpha_{i,n} \in \mathbb{R}$, $T_{i,n}, T'_{i,n} > 0$. We have

$$\begin{aligned} \left| \int_0^T b_n(t)\, a_2\left(x, \frac{x}{\varepsilon}, \xi, \omega\right) g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt \right| &= \left| a_2\left(x, \frac{x}{\varepsilon}, \xi, \omega\right) \right| \left| \int_0^T b_n(t) g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt \right| \\ &\leq ||a_2||_{L^\infty} \left| \int_0^T b_n(t) g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt \right| \end{aligned}$$

and the last term in the right-hand side vanishes as $\varepsilon \to 0$ in $L^1_{loc}(\mathbb{R}^N_x, L^1(\mathbb{R}^N \times \Omega, \mu_c))$ by linearity. Moreover,

$$\begin{aligned} \left| \int_0^T a_1(t)\, a_2\left(x, \frac{x}{\varepsilon}, \xi, \omega\right) g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt \right| &\leq \left| \int_0^T b_n(t)\, a_2\left(x, \frac{x}{\varepsilon}, \xi, \omega\right) g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) dt \right| \\ &\quad + \frac{1}{n} ||a_2||_{L^\infty} ||f_0||_{L^\infty}. \end{aligned}$$

Thus the result holds for $a = a_1(t) a_2(y, \xi, \omega)$, with a_1, $a_2 \in L^\infty$ arbitrary.

For a arbitrary, take a sequence a_δ with $\delta > 0$, converging to a in L^1_{loc}, and such that

$$a_\delta = \sum_{k=0}^{n_\delta} a_1^\delta(t) a_2^\delta(x, y, \xi, \omega).$$

with a_1^δ, a_2^δ in L^∞. The property is known for a_δ, and it is thus easily deduced for a using arguments similar to the ones developed above.

3.2 Second case : $f_0 = f_0(x)$

Unlike the preceding subsection, we now focus on the case when f_0 only depends on the macroscopic variable x. In order to simplify the analysis, we assume that $f_0 \in W^{1,\infty}(\mathbb{R}^N_x)$ (the case when f_0 is not smooth in x will be treated in the next subsection). In that case,

$$f^\varepsilon(t, x, \xi, \omega) = f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)\right).$$

Hence we have to investigate the behavior as $\varepsilon \to 0$ of

$$\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right).$$

We prove the following

Lemma 3.1. *Let $T > 0$ arbitrary. As ε vanishes,*

$$\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) \to 0$$

in $L^\infty((0,T) \times \mathbb{R}^N_x; L^1(\mathbb{R}^N_\xi \times \Omega, \mu_c))$.

Proof. Let us write, for $t > 0$

$$\begin{aligned}
\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) &= \varepsilon \int_0^{\frac{t}{\varepsilon}} \dot{Y}\left(s, \frac{x}{\varepsilon}, \xi, \omega\right) ds + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) \\
&= -t\frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}} \Xi\left(s, \frac{x}{\varepsilon}, \xi, \omega\right) ds + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) \\
&= -t\left\{\frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}} \hat{\xi}\left(T_s(\xi, \tau_{\frac{x}{\varepsilon}}\omega)\right) ds - \xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right\}.
\end{aligned}$$

Let $0 < \alpha < T$ arbitrary. For $\alpha \leq t \leq T$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N_\xi \times \Omega} \left|\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right| d\mu_c(\xi, \omega) \\
= \; & t\int_{\mathbb{R}^N_\xi \times \Omega} \left|\frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}} \hat{\xi}\left(T_s(\xi, \omega)\right) ds - \xi^\sharp(0, \xi, \omega)\right| d\mu_c(\xi, \omega) \\
\leq \; & T \sup_{\tau \geq \frac{\alpha}{\varepsilon}} \left\|\frac{1}{\tau}\int_0^\tau \hat{\xi}\left(T_s(\xi, \omega)\right) ds - \xi^\sharp(0, \xi, \omega)\right\|_{L^1(\mathbb{R}^N \times \Omega, \mu_c)}
\end{aligned}$$

and the upper-bound vanishes as $\varepsilon \to 0$ for any $\alpha > 0$ thanks to corollary 2.1. Notice that the upper-bound does not depend on x, hence the convergence holds in $L^\infty(\mathbb{R}^N_x; L^1(\mu_c))$.

We now have to investigate what happens when t is close to 0; notice that

$$\sup_{x \in \mathbb{R}^N} \left\|\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right\|_{L^1(\mathbb{R}^N \times \Omega, \mu_c)} \leq C_0 \tag{10}$$

where the constant C_0 only depends on N and c. Indeed, if $\mathcal{H}(\xi, \omega) \leq c$, then for all $t \geq 0$,

$$\frac{1}{2}\left|\Xi(t, 0, \xi, \omega)\right|^2 \leq \mathcal{H}(T_t(\xi, \omega)) = \mathcal{H}(\xi, \omega) \leq c.$$

Thus, if $\mathcal{H}(\xi, \omega) \leq c$, then

$$\xi^\sharp(0, \xi, \omega) \leq \sqrt{2cN}.$$

Thus inequality (10) holds with $C_0 = \sqrt{2cN}$.

Similarly, for all $t \geq 0$,

$$\sup_{x \in \mathbb{R}^N} \left\|\hat{\xi}\left(T_s(\xi, \tau_{\frac{x}{\varepsilon}}\omega)\right)\right\|_{L^1(\mathbb{R}^N \times \Omega, \mu_c)} \leq C_0.$$

Hence, if $0 \leq t \leq \alpha$, we have

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N_\xi \times \Omega} \left|\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + \xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right| d\mu_c(\xi, \omega) \leq 2\alpha C_0.$$

Eventually,

$$\left\|\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)-x+t\xi^{\sharp}\left(\frac{x}{\varepsilon},\xi,\omega\right)\right\|_{L^{\infty}((0,T)\times\mathbb{R}^N;L^1(\mu_c))}\leq$$
$$\leq\inf_{0<\alpha<T}\left\{2C_0\alpha+T\sup_{\tau\geq\frac{\alpha}{\varepsilon}}\left\|\frac{1}{\tau}\int_0^{\tau}\hat{\xi}\left(T_s(\xi,\omega)\right)\,ds-\xi^{\sharp}\left(0,\xi,\omega\right)\right\|_{L^1(\mu_c)}\right\}$$

and the lemma is proved. □

We easily deduce that theorem 1 is true when $f_0\in W^{1,\infty}(\mathbb{R}^N)$ with

$$f(t,x,y,\xi,\omega):=f_0(x-t\xi^{\sharp}(y,\xi,\omega)),\quad g=0,$$
$$r^{\varepsilon}(t,x,\xi,\omega):=f^{\varepsilon}(t,x,\xi,\omega)-f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right)$$

and it is easily checked that f satisfies $\mathbb{P}(f)=f$, $f(t=0)=\mathbb{P}(f_0)=f_0$ (since f_0 is independent of y and ξ), and that f is a solution of the evolution equation (4).

3.3 Third case : f_0 arbitrary

We now tackle the case of an arbitrary stationary function $f_0\in L^1_{\text{loc}}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi,L^{\infty}(\mathbb{R}^N_y\times\Omega))$. We begin with the case when

$$f_0(x,y,\xi,\omega)=a(x)b(y,\xi,\omega),$$

with $a\in W^{1,\infty}(\mathbb{R}^N)$ and $b\in L^{\infty}(\mathbb{R}^N_y\times\mathbb{R}^N_\xi\times\Omega)\cap\mathcal{C}^1(\mathbb{R}^N_y\times\mathbb{R}^N_\xi,L^{\infty}(\Omega))$, b stationary. This case follows directly from the two first subsections. Indeed, let

$$f(t,x,y,\xi,\omega)=a(x-t\xi^{\sharp}(y,\xi,\omega))\,\mathbb{P}(b)(y,\xi,\omega),$$

and

$$g(t,x;\tau,y,\xi,\omega)=a(x-t\xi^{\sharp}(y,\xi,\omega))\;(b-\mathbb{P}(b))\,(T_\tau(y,\xi,\omega)).$$

It is already known that f and g satisfy (4), that $f(t,x,\cdot)\in\mathbb{K}$, and that g satisfies (3) thanks to the preceding subsections and the fact that $\xi^{\sharp}(y,\xi,\omega)$ is invariant by the Hamiltonian flow (Y,Ξ). Notice that it is capital here that the coefficient $\xi^{\sharp}(y,\xi,\omega)$ in the transport equation (4) belongs to $\mathbb{K}$.

There remains to check that $g(t,x;\tau,\cdot)\in\mathbb{K}^{\perp}$, that the remainder r^{ε} goes to 0 strongly in L^1_{loc} and that $g(t,x;t/\varepsilon,x/\varepsilon,\xi,\omega)$ goes weakly to 0 in the sense of theorem 1. First, notice that $a(x-t\xi^{\sharp}(y,\xi,\omega))\in\mathbb{K}$ and $(b-\mathbb{P}(b))\,(T_\tau(y,\xi,\omega))\in\mathbb{K}^{\perp}$. Thus, $a(x-t\xi^{\sharp})\mathbb{P}(b)=\mathbb{P}(a(x-t\xi^{\sharp})b)$ almost everywhere (because $a(x-t\xi^{\sharp}(0,\xi,\omega))$ is invariant by the semi-group T_τ), and consequently

$$g(t,x;\tau,y,\xi,\omega)=\left[a(x-t\xi^{\sharp})b-\mathbb{P}\left(a(x-t\xi^{\sharp})b\right)\right](T_\tau(\xi,\tau_y\omega)).$$

Hence $g(t,x;\tau,\cdot)\in\mathbb{K}^{\perp}$ a.e.

Then, setting

$$r^{\varepsilon}(t,x,\xi,\omega)=f^{\varepsilon}(t,x,\xi,\omega)-f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right)-g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),$$

we have to prove that r^{ε} goes to 0 strongly in L^1_{loc}. We compute the difference

$$\begin{aligned}
&f^{\varepsilon}(t,x,\xi,\omega)-f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right)-g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\\
=\;&a\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right)b\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\\
&-a\left(x-t\xi^{\sharp}\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)\;\mathbb{P}(b)\left(\frac{x}{\varepsilon},\xi,\omega\right)\\
&-a\left(x-t\xi^{\sharp}\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)[b-\mathbb{P}(b)]\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\\
=\;&\left[a\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right)-a\left(x-t\xi^{\sharp}\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)\right]b\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)
\end{aligned}$$

The right-hand side of the above equality is bounded by

$$||a||_{W^{1,\infty}}||b||_{L^\infty}\left|\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)-x+t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right|$$

and thus converges to 0 as $\varepsilon\to 0$ in $L^\infty((0,T)\times\mathbb{R}^N_x;L^1(\mathbb{R}^N_\xi\times\Omega,\mu_c))$ according to the second subsection.

Moreover, it is easily proved that as $\varepsilon\to 0$,

$$\int_0^T g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt\to 0$$

strongly in $L^1_{\text{loc}}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi,L^1(\Omega))$ thanks to remark 3.1. Hence theorem 1 is proved in that case.

The general case now follows from a density result and a contraction property, which are stated in the two lemmas below. We first explain how the general case can be deduced from the lemmas, and then we prove the lemmas.

The first lemma states that linear combinations of functions of the type $a(x)b(y,\xi,\omega)$, with a and b smooth, are dense in $L^1_{\text{loc}}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi,L^\infty(\mathbb{R}^N_y\times\Omega))$.

Lemma 3.2. *Let $f_0\in L^1_{loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi,L^\infty(\mathbb{R}^N_y\times\Omega))$ arbitrary, and set $F_0(x,\xi,\omega):=f_0(x,0,\xi,\omega)$. Let $R,R'>0$ arbitrary.*

There exists a sequence of functions $F_n\in L^1(\mathbb{R}^N_x\times\mathbb{R}^N_\xi\times\Omega)$ such that

- *$F_n\to F_0$ as $n\to\infty$ in $L^1(B_R\times B_{R'}\times\Omega)$;*
- *For all $n\in\mathbb{N}$, there exist an integer $N_n\in\mathbb{N}$ and functions $a_i^n\in\mathcal{C}^1\cap W^{1,\infty}(\mathbb{R}^N)$, $b_i^n\in L^\infty(\mathbb{R}^N_\xi\times\Omega)$, $1\le i\le N_n$, such that almost surely in ω, for all $(x,\xi)\in\mathbb{R}^{2N}$*

$$F_n(x,\xi,\omega)=\sum_{i=1}^{N_n}a_i^n(x)\,b_i^n(\xi,\omega)\ ;$$

- *For all $n\in\mathbb{N}$, for $1\le i\le N_n$, the function*

$$(y,\xi,\omega)\mapsto b_i^n(\xi,\tau_y\omega)$$

belongs to $\mathcal{C}^1(\mathbb{R}^N_y\times\mathbb{R}^N_\xi,L^\infty(\Omega))$.

The second lemma states a contraction property for equation (1) and for equation (4).

Lemma 3.3. *Let $g_0\in L^1_{loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi,L^\infty(\mathbb{R}^N_y\times\Omega))$ be a stationary admissible data for (1). Let g^ε be the solution of (1) with initial data $g_0\,(x,x/\varepsilon,\xi,\omega)$. Then for all $R,R',T>0$, for all $t\in[0,T]$,*

$$E\left[\int_{x\in B_R,\ \xi\in B_{R'}}|g^\varepsilon(t,x,\xi,\omega)|\ dx\,d\xi\right]\le||G_0||_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega,dx\,d\mu_{c_{R'}}(\xi,\omega))}\tag{11}$$

where

$$G_0(x,\xi,\omega):=g_0(x,0,\xi,\omega),$$
$$C_{R,T,R'}:=\left\{x\in\mathbb{R}^N,\ |x|\le R+T\sqrt{{R'}^2+2u_{max}}\right\},$$
$$c_{R'}:=\frac{1}{2}{R'}^2+u_{max}.$$

Similarly, if g is a solution of (4) with initial data $g_0\in L^1_{loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi,L^\infty(\mathbb{R}^N_y\times\Omega))$, then

$$\int_{x\le R}|g(t,x,y,\xi,\omega)|\ dx\le\int_{x\le R+T\sqrt{\xi^2+2u_{max}}}|g_0(x,y,\xi,\omega)|\ dx.\tag{12}$$

And if h is solution of (3) with initial data $h_0\in L^1_{loc}(\mathbb{R}^N_\xi,L^\infty(\mathbb{R}^N_y\times\Omega))$, then for all $y\in\mathbb{R}^N,\ \tau\ge 0$,

$$E\left[\int_{\xi\in B_{R'}}|h(\tau,y,\xi,\omega)|\ d\xi\right]\le\int_{\mathbb{R}^N_\xi\times\Omega}|h_0|(y,\xi\,\omega)\ d\mu_{c_{R'}}(\xi,\omega).\tag{13}$$

We postpone the proofs of lemmas 3.2 and 3.3.

Now, let $R, R' > 0$ abritrary, and let F_n be a sequence converging to F_0 in $L^1(B_R \times B_{R'} \times \Omega)$ as in lemma 3.2. Assume that f_0 is an admissible initial data for (1), and let f_n^ε (resp. f^ε) be the solution of (1) with initial data $F_n\left(x, \xi, \tau_{\frac{x}{\varepsilon}}\omega\right)$ (resp. $F_0\left(x, \xi, \tau_{\frac{x}{\varepsilon}}\omega\right)$), and let $f_n = f_n(t, x, y, \xi, \omega)$, $g_n = g_n(t, x; \tau, y, \xi, \omega)$ be the functions associated to f_n^ε by theorem 1 for all n.

Let $f(t, x, y, \xi, \omega)$, $g(t, x; \tau, y, \xi, \omega)$ be the solutions of the system

$$\begin{gathered}
\mathbb{P}(f) = f, \quad \mathbb{P}(g) = 0,\\
\partial_t \begin{pmatrix} f \\ g \end{pmatrix} + \xi^\sharp(y, \xi, \omega) \cdot \nabla_x \begin{pmatrix} f \\ g \end{pmatrix} = 0,\\
\partial_\tau g + \xi \cdot \nabla_y g - \nabla_y u(y, \omega) \cdot \nabla_\xi g = 0,\\
f(t=0) = \mathbb{P}(f_0), \quad g(t=0, x; \tau=0, y, \xi, \omega) = \left[f_0 - \mathbb{P}(f_0)\right](x, y, \xi, \omega).
\end{gathered}$$

We have already proved that f_n, g_n satisfy the above system. We denote by $\bar{F}_0$, $\bar{F}_n$, the functions associated to F_0, F_n respectively by corollary 2.1, so that $\mathbb{P}(f_0)(x, y, \xi, \omega) = \bar{F}_0(x, \xi, \tau_y\omega)$, and $f_n(t = 0, x, y, \xi, \omega) = \bar{F}_n(x, \xi, \tau_y\omega)$, $g_n(t = 0, x, \tau = 0, y, \xi, \omega) = (F_n - \bar{F}_n)(x, \xi, \tau_y\omega)$.

Notice that if $G \in L^1(\mathbb{R}^N_\xi \times \Omega, \mu_c)$, for some $c > 0$, then μ_c almost everywhere

$$\begin{aligned}
|\bar{G}(\xi, \omega)| &= \lim_{T\to\infty} \frac{1}{T} \left| \int_0^T G(T_t(\xi, \omega))\, dt \right|\\
&\leq \lim_{T\to\infty} \frac{1}{T} \int_0^T |G(T_t(\xi, \omega))|\, dt = \overline{|G|}\,(y, \xi),
\end{aligned}$$

and thus, according to property (7),

$$\begin{aligned}
||\bar{G}||_{L^1(\mathbb{R}^N_\xi \times \Omega, \mu_c)} &\leq \int_{\mathbb{R}^N \times \Omega} \overline{|G|}\,(y, \xi)\, d\mu_c(y, \xi)\\
&= \int_{\mathbb{R}^N \times \Omega} |G|(y, \xi)\, d\mu_c(y, \xi) = ||G||_{L^1(\mathbb{R}^N_\xi \times \Omega, \mu_c)}.
\end{aligned}$$

Consequently, setting $c := c_{R'}$, we have

$$\begin{aligned}
||f(t, \cdot, y, \xi, \omega) - f_n(t, \cdot, y, \xi, \omega)||_{L^1(B_R)} &\leq \left|\left|\bar{F}_0(\cdot, \xi, \tau_y\omega) - \bar{F}_n(\cdot, \xi, \tau_y\omega)\right|\right|_{L^1(C_{R,T,\xi})}\\
||f(t, x, y, \xi, \omega) - f_n(t, x, y, \xi, \omega)||_{L^1(B_R \times B_{R'} \times \Omega)} &\leq \left|\left|\overline{F_0 - F_n}\right|\right|_{L^1(C_{R,T,R'} \times \mathbb{R}^N_\xi \times \Omega, dx d\mu_c(\xi,\omega))}\\
&\leq ||F_0 - F_n||_{L^1(C_{R,T,R'} \times \mathbb{R}^N_\xi \times \Omega, dx d\mu_c(\xi,\omega))}.
\end{aligned}$$

And similarly, using (11), (12), (13),

$$\begin{gathered}
||f^\varepsilon(t, x, \xi, \omega) - f_n^\varepsilon(t, x, \xi, \omega)||_{L^1(B_R \times B_{R'} \times \Omega)} \leq ||F_0 - F_n||_{L^1(C_{R,T,R'} \times \mathbb{R}^N_\xi \times \Omega, dx\, d\mu_c(\xi,\omega))},\\
||g(t, x; \tau, y, \xi, \omega) - g_n(t, x; \tau, y, \xi, \omega)||_{L^1(B_R \times B_{R'} \times \Omega)} \leq 2\, ||F_0 - F_n||_{L^1(C_{R,T,R'} \times \mathbb{R}^N_\xi \times \Omega, dx d\mu_c(\xi,\omega))}.
\end{gathered}$$

The above inequalities are true for all $t \in [0, T]$ and for all $\tau \geq 0$.

Set

$$r^\varepsilon(t, x, \xi, \omega) := f^\varepsilon(t, x, \xi, \omega) - f\left(t, x, \frac{x}{\varepsilon}, \xi, \omega\right) - g\left(t, x; \frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right).$$

Then for all $t \in [0, T]$, for all $n \in \mathbb{N}$,

$$\begin{aligned}
||r^\varepsilon(t)||_{L^1(B_R \times B_{R'} \times \Omega)} \leq\ & ||f^\varepsilon(t) - f_n^\varepsilon(t)||_{L^1(B_R \times B_{R'} \times \Omega)}\\
&+ ||f(t) - f_n(t)||_{L^\infty(\mathbb{R}^N_y; L^1(B_R \times B_{R'} \times \Omega))}\\
&+ \left|\left| g\left(t; \frac{t}{\varepsilon}\right) - g_n\left(t; \frac{t}{\varepsilon}\right) \right|\right|_{L^\infty(\mathbb{R}^N_y; L^1(B_R \times B_{R'} \times \Omega))}\\
&+ ||r_n^\varepsilon(t)||_{L^1(B_R \times B_{R'} \times \Omega)}\\
\leq\ & 4\, ||F_0 - F_n||_{L^1(C_{R,T,R'} \times \mathbb{R}^N_\xi \times \Omega, dx d\mu_c(\xi,\omega))} + ||r_n^\varepsilon(t)||_{L^1(B_R \times B_{R'} \times \Omega)}.
\end{aligned}$$

Thus $r^\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^\infty([0,\infty); L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi; L^1(\Omega))$.

There only remains to check that $\int_0^T g(t,x;t/\varepsilon,x/\varepsilon,\xi,\omega)\,dt$ goes strongly to 0 in L^1_{loc} norm as ε vanishes; this result follows immediately from the same property for g_n and the above inequalities. Therefore, we skip its proof. □

We now tackle the proofs of lemmas 3.2 and 3.3.

Proof of Lemma 3.2. We use the results of chapter 2 in [5]. Since $F_0 \in L^1(B_R \times B_{R'+1}, L^\infty(\Omega))$, $F_0 \in L^1(B_R, L^1(B_{R'+1} \times \Omega))$. Thus, there exists a sequence of functions $(\tilde{F}_n)_{n\in\mathbb{N}}$ such that $\tilde{F}_n \to F_0$ in $L^1(B_R \times B_{R'+1} \times \Omega)$, and for all $n \in \mathbb{N}$,

$$\tilde{F}_n = \sum_{i=1}^{N_n} \mathbf{1}_{A_{i,n}}(x)\phi_{i,n}(\xi,\omega),$$

where $N_n \in \mathbb{N}$, and :

- for all $n \in \mathbb{N}$, $1 \le i \le N_n$, $A_{i,n} \subset B_R$ is a measurable set;
- for all $n \in \mathbb{N}$, $1 \le i \le N_n$, $\phi_{i,n} \in L^1(B_{R'+1} \times \Omega)$.

We shall explain later why we have chosen an approximating sequence in $L^1(B_R \times B_{R'+1} \times \Omega)$ rather than $L^1(B_R \times B_{R'} \times \Omega)$.

The idea is to replace $\tilde{F}_n$ by a function F_n having the same structure, but in which the functions $1_{A_{i,n}}$, $\phi_{i,n}$ have been regularized. Hence we consider a mollifier $\rho \in \mathcal{D}(\mathbb{R}^N)$ such that $\rho \ge 0$ and $\int_{\mathbb{R}^N} \rho = 1$. For $k \in \mathbb{N}$, set

$$\rho_k(x) = k^N \rho(kx), \quad x \in \mathbb{R}^N.$$

The regularization by convolution in the variable x is standard : $\mathbf{1}_{A_{i,n}}(x)$ is replaced by

$$\mathbf{1}_{A_{i,n}} *_x \rho_k(x) = \int_{\mathbb{R}^N} \mathbf{1}_{A_{i,n}}(y)\rho_k(x-y).$$

Concerning the regularization of the functions $\phi_{i,n}$, we first truncate $\phi_{i,n}$ in order to obtain a function in L^∞. For $M > 0$, set

$$\phi^M_{i,n} := \text{sgn}(\phi_{i,n}) \inf(|\phi_{i,n}|, M).$$

Then $\phi^M_{i,n} \in L^\infty(B_{R'+1} \times \Omega)$, and $\phi^M_{i,n}$ converges towards $\phi_{i,n}$ as $M \to \infty$ in $L^1(B_{R'+1} \times \Omega)$. Thus we work with $\phi^M_{i,n}$ instead of $\phi_{i,n}$, and we drop the superscript M in order to avoid too heavy notations.

Now, for $k \in \mathbb{N}$, we set, for $y \in \mathbb{R}^N$, $\xi \in B_{R'}$,

$$\varphi_{i,n}(y,\xi,\omega) = \phi_{i,n}(\xi,\tau_y\omega),$$
$$\varphi^k_{i,n}(y,\xi,\omega) = \int_{\mathbb{R}^{2N}} \phi_{i,n}(\xi',\tau_{y'}\omega)\,\rho_k(y-y')\rho_k(\xi-\xi')\,dy'\,d\xi',$$
$$\phi^k_{i,n}(\xi,\omega) = \varphi^k_{i,n}(0,\xi,\omega).$$

This is the part where it is convenient to have $\phi_{i,n} \in L^1(B_{R'+1},\Omega)$. Indeed, if $\xi \in B_{R'}$ and $|\xi - \xi'| \le 1$, then $\xi' \in B_{R'+1}$; thus the convolution is well-defined on $\mathbb{R}^N \times B_{R'}$ as long as $\phi_{i,n} \in L^1(B_{R'+1},\Omega)$.

The function $\varphi^k_{i,n}$ belongs to $\mathcal{C}^1_b(\mathbb{R}^N \times B_{R'}, L^\infty(\Omega))$, where $\mathcal{C}^1_b$ denotes the space of $\mathcal{C}^1$ bounded functions with bounded derivatives. Moreover, it is easily checked that $\varphi^k_{i,n}$ is stationary, and $\varphi^k_{i,n}(\cdot,\omega)$ converges towards $\varphi_{i,n}(\omega)$ in $L^1_{\text{loc}}(\mathbb{R}^N_y, L^1(B_{R'}))$, almost surely in ω. And if K is any compact set in $\mathbb{R}^N$, then there exists a compact set K' such that $K \subset K'$ and almost surely in ω,

$$||\varphi^k_{i,n}(\cdot,\omega) - \varphi_{i,n}(\cdot,\omega)||_{L^1(K\times B_{R'})} \le 2||\varphi_{i,n}(\cdot,\omega)||_{L^1(K'\times B_{R'})}.$$

Since $||\varphi_{i,n}(\cdot,\omega)||_{L^1(K'\times B_{R'})}$ belongs to $L^1(\Omega)$, using Lebesgue's dominated convergence theorem, we deduce that $\varphi^k_{i,n}$ converges towards $\varphi_{i,n}$ in $L^1(K \times B_{R'} \times \Omega)$, for every compact set $K \subset \mathbb{R}^N$. Thus $\phi^k_{i,n}$ converges towards $\phi_{i,n}$ in $L^1(B_{R'} \times \Omega)$ as $k \to \infty$, due to the invariance of the measure P with respect to the transformation group τ_y.

We set

$$\tilde{F}_{n,k}(x,\xi,\omega) := \sum_{i=1}^{N_n} \mathbf{1}_{A_{i,n}} *_x \rho_k(x)\ \phi_{i,n}^k(\xi,\omega).$$

Then $\tilde{F}_{n,k}$ converges towards $\tilde{F}_n$ as $k\to\infty$ in $L^1(B_R\times B_{R'}\times\Omega)$. Thus there exists an integer k_n such that

$$||\tilde{F}_{n,k_n} - \tilde{F}_n||_{L^1(B_R\times B_{R'}\times\Omega)} \le \frac{1}{n}.$$

Set $F_n := \tilde{F}_{n,k_n}$. Then F_n converges towards F_0 in L^1 as $n\to\infty$, and thus the lemma is true.

□

Proof of Lemma 3.3. We first prove the lemma when $g_0 \in \mathcal{C}^1(\mathbb{R}^N_x\times\mathbb{R}^N_y\times\mathbb{R}^N_\xi, L^\infty(\Omega))$. In that case, let us recall that

$$\begin{aligned} g^\varepsilon(t,x,\xi,\omega) &= g_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\\ &= g_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)+x, Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right), \Xi\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right),\tau_{\frac{x}{\varepsilon}}\omega\right)\\ &= G_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)+x, T_{\frac{t}{\varepsilon}}\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right), \end{aligned}$$

where $G_0(z,\xi,\omega) := g_0(z,0,\xi,\omega)$ for all $(z,\xi,\omega)\in\mathbb{R}^N\times\mathbb{R}^N\times\Omega$, and the semi-group $(T_t)_{t\ge0}$ was defined in section 2 (see lemma 2.1). Moreover, since

$$\frac{1}{2}\left|\Xi(t,y,\xi,\omega)\right|^2 + u\left(Y(t,y,\xi,\omega)\right) = \frac{1}{2}|\xi|^2 + u(y,\omega)$$

we have

$$|\Xi(t,y,\xi,\omega)| \le \sqrt{|\xi|^2+2u(y,\omega)} \le \sqrt{|\xi|^2+2u_{\max}}$$

and almost surely in ω,

$$\left|\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\omega\right)\right| \le t\sqrt{|\xi|^2+2u_{\max}}.$$

Additionnally, we have, for all $R'>0$, for almost every $\xi\in\mathbb{R}^N$ and almost surely in ω,

$$\mathbf{1}_{|\xi|\le R'} \le \mathbf{1}_{\mathcal{H}(\xi,\omega)\le\frac{1}{2}R'^2+u_{\max}}.$$

(Remember that $\mathcal{H}(\xi,\omega)=\frac{1}{2}|\xi|^2+u(0,\omega)$). Thus, setting $c_{R'} := \frac{1}{2}{R'}^2+u_{\max}$, we have

$$\begin{aligned} &E\left[\int_{x\in B_R,\ \xi\in B_{R'}} |g^\varepsilon(t,x,\xi,\omega)|\ dx\, d\xi\right]\\ =\ &\int_{x\in B_R,\ \xi\in B_{R'}} E\left[\left|G_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)+x, T_{\frac{t}{\varepsilon}}\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right)\right|\right]\ dx\, d\xi\\ =\ &\int_{x\in B_R,\ \xi\in B_{R'}} E\left[\left|G_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\omega\right)+x, T_{\frac{t}{\varepsilon}}(\xi,\omega)\right)\right|\right]\ dx\, d\xi\\ =\ &E\left[\int_{\xi\in B_{R'}}\left(\int_{x\in B_R}\left|G_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\omega\right)+x, T_{\frac{t}{\varepsilon}}(\xi,\omega)\right)\right|\ dx\right)\, d\xi\right]\\ \le\ &E\left[\int_{\xi\in B_{R'}}\left(\int_{|z|\le R+T\sqrt{R'^2+2u_{\max}}}\left|G_0\left(z, T_{\frac{t}{\varepsilon}}(\xi,\omega)\right)\right|\ dz\right)\, d\xi\right]\\ \le\ &\int_{|z|\le R+T\sqrt{R'^2+2u_{\max}}}\int_{\mathbb{R}^N\times\Omega}\left|G_0\left(z, T_{\frac{t}{\varepsilon}}(\xi,\omega)\right)\right|\ d\mu_{c_{R'}}(\xi,\omega)\, dz\\ =\ &\int_{|z|\le R+T\sqrt{R'^2+2u_{\max}}}\int_{\mathbb{R}^N\times\Omega}\left|G_0(z,\xi,\omega)\right|\ d\mu_{c_{R'}}(\xi,\omega)\, dz\\ =\ &||G_0||_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega, dx\, d\mu_{c_{R'}}(\xi,\omega))}. \end{aligned}$$

Now, if g_0 is an arbitrary admissible data in $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_y \times \mathbb{R}^N_\xi, L^\infty(\Omega))$, we choose a sequence G_n approaching G_0 as in lemma 3.2 (R and R' are fixed). Then inequality (11) is true for g^ε_n for all $n \in N$, and for $g^\varepsilon_n - g^\varepsilon_m$ for all $n, m \in N$. Thus the sequence g^ε_n is a Cauchy sequence in $L^\infty_{\text{loc}}((0,\infty), L^1(B_R \times B_{R'} \times \Omega))$. Thus it converges strongly towards a solution of the transport equation (1). Thanks to a uniqueness result for the transport equation (1), the limit of the sequence g^ε_n as $n \to \infty$ is g^ε. There only remains to pass to the limit as $n \to \infty$ in the inequality (11) written for g^ε_n and G^n_0.

The proof of inequalities (12) and (13) go along the same lines.

□

4 The integrable case

In this section, we treat independently the periodic and the stationary ergodic case. Indeed, some results of the periodic case are no longer true in the stationary ergodic setting, and the results which do remain valid are not proved with the same tools.

Let us make precise what we mean about "integrable case" : in the periodic case, we take a function $u(y)$ which has the form

$$u(y) = \sum_{i=1}^N u_i(y_i), \tag{14}$$

where each function u_i is periodic with period 1 ($1 \leq i \leq N$). The Hamiltonian $H(y,\xi)$ can be written

$$H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y) = \sum_{i=1}^N H_i(y_i,\xi_i)$$

where $H_i(y_i,\xi_i) = \frac{1}{2}|\xi_i|^2 + u_i(y_i)$ ($1 \leq i \leq N$). And the Hamiltonian system (5) becomes

$$\begin{cases} \dot{Y}_i = -\Xi_i, \\ \dot{\Xi}_i = u_i'(Y_i), \\ Y_i(t=0) = y_i, \quad \Xi_i(t=0) = \xi_i. \end{cases} \tag{15}$$

Thus it is enough to investigate the behavior of each one-dimensional Hamiltonian system (15) individually, and for most calculations, we can assume without loss of generality that $N = 1$, and we drop all indices i. However, for the calculation of the projection $\mathbb{P}$, a more thorough discussion will be needed, and we will come back to the case where $N > 1$ in the corresponding paragraph.

In the stationary ergodic setting, expression (14) can be transposed in the following way : assume that $\Omega = \Pi_{i=1}^N \Omega_i$, where each Ω_i is a probability space, and assume that for $1 \leq i \leq N$, an ergodic group transformation, denoted by $(\tau_{i,y})_{y\in\mathbb{R}}$, acts on each Ω_i.

Then for $\omega = (\omega_1, \cdots, \omega_N) \in \Omega$, and $y = (y_1, \cdots, y_N) \in \mathbb{R}^N$, we set $\tau_y\omega := (\tau_{1,y_1}\omega_1, \cdots, \tau_{N,y_N}\omega_N)$. And we assume that the function u can be written

$$u(y,\omega) = \sum_{i=1}^N U_i\left(\tau_{i,y_i}\omega_i\right),$$

where $U_i \in L^\infty(\Omega_i)$ for all $1 \leq i \leq N$. The same remarks as in the periodic case can be made, and thus we will only consider the case $N = 1$; note that in the stationary ergodic case, we are unable to compute the projection $\mathbb{P}$ when $N > 1$.

4.1 Periodic setting

The goal of this subsection is to give another proof of the results of K. Hamdache and E. Frénod in [10], based on the study of the system

$$\begin{cases} \dot{Y} = -\Xi, \\ \dot{\Xi} = u'(Y), \\ Y(t=0) = y, \quad \Xi = \xi, \quad y \in \mathbb{R},\ \xi \in \mathbb{R}. \end{cases} \tag{16}$$

The Hamiltonian $H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y)$ is constant along the trajectories of the system (16), so that

$$\frac{1}{2}|\Xi(t,y,\xi)|^2 + u(Y(t,y,\xi)) = H(y,\xi).$$

We now fix $y, \xi \in \mathbb{R}^N$. Without any loss of generality, we assume $y \in [-\frac{1}{2}, \frac{1}{2})$, and we set $\mathcal{E} := H(y,\xi)$. The above equation describes the movement of a single particle in a periodic potential u, with $0 \le u \le u_{\max}$. It is well-known that there are two kinds of behavior, depending on the value of the energy $\mathcal{E}$: if $\mathcal{E} < u_{\max}$, the particle is "trapped" in a well of potential around y, and $Y(t)$ remains bounded as $t \to \infty$. In that case, the trajectories in the phase space are closed curves. If $\mathcal{E} > u_{\max}$, the trajectory of the particle is unconstrained and $|Y(t)| \to \infty$ as $t \to \infty$. We study more precisely these two cases and their consequences on the expression of the projection $\mathbb{P}$ in the following subsections ; we refer for instance to [3] for further calculations and results about Hamiltonian dynamics and ordinary differential equations in general.

4.1.1 Expression of $\xi^\sharp(y,\xi)$

We begin with the case when $H(y,\xi) < u_{\max}$. In that case, $u(y) \le H(y,\xi) < u_{\max}$. By continuity of the potential u, there exists $y_- < y$ and $y_+ > y$ such that $H(y,\xi) < u(y_\pm) < u_{\max}$, and the periodicity of u allows us to choose $y_\pm$ such that $|y_+ - y_-| < 1$. Then $y_- < Y(t,y,\xi) < y_+$ for all $t \ge 0$. Indeed, assume that there exists $t > 0$ such that $Y(t,y,\xi) \ge y_+ > y = Y(t=0,y,\xi)$. Since the trajectory Y is continuous in time, there exists $0 < t_0 \le t$ such that $Y(t=t_0,y,\xi) = y_+$, which is absurd since

$$H(Y(t_0,y,\xi),\Xi(t_0,y,\xi)) = H(y,\xi) \ge u(Y(t_0,y,\xi)) = u(y_+) > H(y,\xi).$$

Thus $Y(t,y,\xi)$ is bounded. Since

$$\xi^\sharp(y,\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Xi(t,y,\xi)\,dt = -\lim_{T\to\infty}\frac{1}{T}\int_0^T \dot{Y}(t,y,\xi)\,dt = \lim_{T\to\infty}\frac{y - Y(T,y,\xi)}{T}$$

we deduce that $\xi^\sharp(y,\xi) = 0$ for all y,ξ such that $H(y,\xi) < u_{\max}$.

We now study the case $H(y,\xi) > u_{\max}$. Since

$$|\dot{Y}(t,y,\xi)|^2 = 2\left(H(y,\xi) - u(Y(t,y,\xi))\right) \ge 2(H(y,\xi) - u_{\max}) > 0,$$

we deduce that $\dot{Y}$ does not vanish for $t \ge 0$. Consequently,

$$\Xi(t,y,\xi) = -\dot{Y}(t,y,\xi) = \operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(Y(t,y,\xi))\right)},$$

and since $|Y(t,y,\xi) - y| \ge \sqrt{2\left(H(y,\xi) - u_{\max}\right)}t$, $|Y(t)| \to \infty$ as $t \to \infty$. We immediately deduce that $\Xi(t,y,\xi)$ is periodic in time: indeed, there exists $t_0 > 0$ such that

$$Y(t_0,y,\xi) = y - \operatorname{sgn}(\xi).$$

And

$$\begin{aligned}\Xi(t=t_0,y,\xi) &= \operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(Y(t_0,y,\xi))\right)}\\ &= \operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(y)\right)}\\ &= \xi = \Xi(t=0,y,\xi),\end{aligned}$$

so that for $s \ge 0$,

$$Y(t_0+s,y,\xi) = Y(s,y,\xi) - \operatorname{sgn}(\xi),$$
$$\Xi(t_0+s,y,\xi) = \Xi(s,y,\xi),$$

and Ξ is periodic with period t_0.

Consequently,

$$\xi^\sharp(y,\xi) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Xi(t,y,\xi)\,dt = \frac{1}{t_0}\int_0^{t_0}\Xi(t,y,\xi)\,dt.$$

But

$$\begin{aligned}\int_0^{t_0}\Xi(t,y,\xi)\,dt &= -\int_0^{t_0}\dot{Y}(t,y,\xi)\,dt\\ &= -\left(Y(t_0,y,\xi)-y\right)\\ &= \mathrm{sgn}(\xi).\end{aligned}$$

Thus we only have to compute t_0. With this aim in view, we use the change of variables $s = Y(t)$, with Jacobian $ds = \dot{Y}dt$ (recall that $\dot{Y}(t,y,\xi) = -\mathrm{sgn}(\xi)\sqrt{2\left(H(y,\xi)-u(Y(t,y,\xi))\right)}$), in the formula

$$\begin{aligned}t_0 &= \int_0^{t_0} dt\\ &= \int_{Y(t=0)}^{Y(t_0)}\frac{1}{-\mathrm{sgn}(\xi)\sqrt{2\left(H(y,\xi)-u(s)\right)}}ds\\ &= -\mathrm{sgn}(\xi)\int_y^{y-\mathrm{sgn}(\xi)}\frac{1}{\sqrt{2\left(H(y,\xi)-u(s)\right)}}ds\\ &= \int_0^1\frac{1}{\sqrt{2\left(H(y,\xi)-u(s)\right)}}ds\end{aligned}$$

Eventually, we deduce

$$\xi^\sharp(y,\xi) = \mathrm{sgn}(\xi)\varphi(H(y,\xi)),$$

where

$$\varphi(\mathcal{E}) = \sqrt{2}\mathbf{1}_{\mathcal{E}>u_{\max}}\frac{1}{\left\langle\frac{1}{\sqrt{(\mathcal{E}-u(s))}}\right\rangle}$$

We close this paragraph with a calculation which allows us to express $\xi^\sharp$ in terms of the homogenized Hamiltonian $\bar{H}$. The result we will obtain will be justified in more abstract and theoretical terms in the last subsection, using arguments similar to those of the theory of Aubry-Mather.

First, let us recall the expression of the homogenized Hamiltonian $\bar{H}$ (see [13]) : we have

$$H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y), \quad \text{with}\inf u = 0,\ \sup u = u_{\max},$$

and thus

$$\bar{H}(p) = u_{\max} + \frac{1}{2}\begin{cases}0 & \text{if } p < \left\langle\sqrt{2(u_{\max}-u)}\right\rangle\\ \lambda & \text{if } |p| \geq \left\langle\sqrt{2(u_{\max}-u)}\right\rangle,\end{cases} \quad \text{where } |p| = \left\langle\sqrt{2(u_{\max}-u)+\lambda}\right\rangle.$$

In other words, setting

$$\theta : \begin{array}{ccc}[0,\infty) & \to & [0,\infty)\\ \lambda & \mapsto & \left\langle\sqrt{2(u_{\max}-u)+\lambda}\right\rangle\end{array}$$

we have

$$\bar{H}(p) = u_{\max} + \frac{1}{2}\mathbf{1}_{|p|\geq\theta(0)}\theta^{-1}(|p|).$$

Hence,

$$\bar{H}'(p) = \mathrm{sgn}(p)\frac{1}{2}\mathbf{1}_{|p|>\theta(0)}\frac{1}{\theta'\left(\theta^{-1}(|p|)\right)};$$

and

$$\theta'(\lambda) = \frac{1}{2}\left\langle \frac{1}{\sqrt{2(u_{\max}-u)+\lambda}} \right\rangle,$$
$$\theta^{-1}(|p|) = 2\left(\bar{H}(p) - u_{\max}\right) \quad \forall |p| \geq \theta(0),$$
$$|p| > \theta(0) \iff \bar{H}(p) > u_{\max} \quad \forall p.$$

Gathering all the terms, we are led to

$$\begin{aligned} \bar{H}'(p) &= \operatorname{sgn}(p)\sqrt{2}\mathbf{1}_{\bar{H}(p)>u_{\max}} \frac{1}{\left\langle \frac{1}{\sqrt{\bar{H}(p)-u}} \right\rangle} \\ &= \operatorname{sgn}(p)\varphi\left(\bar{H}(p)\right) \end{aligned}$$

Thus, the final expression is

$$\xi^\sharp(y,\xi) = \bar{H}'(p),$$

where p is such that

$$\bar{H}(p) = H(y,\xi) \vee u_{\max}, \quad \operatorname{sgn}(p) = \operatorname{sgn}(\xi).$$

4.1.2 Expression of the projection $\mathbb{P}$

We also mention here how to find a general expression of the projection $\mathbb{P}$ in the special case $N=1$, and we explain how to generalize this expression in some particular cases when $N>1$. Recall that if $f = f(y,\xi) \in L^1_{\text{loc}}(\mathbb{R}^N_y \times \mathbb{R}^N_\xi)$ is periodic in y, then

$$\mathbb{P}(f)(y,\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt$$

and the limit holds almost everywhere and in $L^1([0,1)\times\mathbb{R}^N, m_c)$, with $dm_c(y,\xi) = \mathbf{1}_{H(y,\xi)\leq c}\,dy\,d\xi$.

We begin with the case $H(y,\xi) > u_{\max}$. We have seen in the previous paragraph that there exists $t_0 > 0$, which depends only on $H(y,\xi)$ such that for all $t>0$, for all $k\in\mathbb{N}$

$$Y(t+k\,t_0,y,\xi) = Y(t,y,\xi) - k\operatorname{sgn}(\xi), \quad \Xi(t+k\,t_0,y,\xi) = \Xi(t,y,\xi).$$

Thus $f(Y(t),\Xi(t))$ is periodic in time with period t_0, and

$$\mathbb{P}(f)(y,\xi) = \frac{1}{t_0}\int_0^{t_0} f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt.$$

We use once again the change of variables $s = Y(t)$, so that

$$\begin{aligned} &\int_0^{t_0} f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt \\ =\ &\int_y^{y-\operatorname{sgn}(\xi)} f\left(s, \operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi)-u(s)\right)}\right) \frac{1}{-\operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi)-u(s)\right)}}\,ds \\ =\ &\left\langle f\left(\cdot, \operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi)-u(\cdot)\right)}\right) \frac{1}{\sqrt{2\left(H(y,\xi)-u(\cdot)\right)}} \right\rangle. \end{aligned}$$

And eventually,

$$\mathbb{P}(f)(y,\xi) = \bar{f}(\operatorname{sgn}(\xi), H(y,\xi)) \tag{17}$$

with

$$\bar{f}(\eta,\mathcal{E}) := \frac{\left\langle f\left(\cdot, \eta\sqrt{2\left(\mathcal{E}-u(\cdot)\right)}\right) \frac{1}{\sqrt{(\mathcal{E}-u(\cdot))}} \right\rangle}{\left\langle \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle} \quad \eta = \pm 1,\ \mathcal{E} > u_{\max}.$$

We now focus on the case $0 < \mathcal{E} < u_{\max}$. In order to simplify the analysis we assume that $\mathcal{E} \notin \{u(y)\,; u \text{ has a local extremum at } y\}$ (this set is finite or countable), and that

$$\forall y \in \mathbb{R}, \quad u'(y) = 0 \Rightarrow u \text{ has a local extremum at } y.$$

In that case, it can be easily proved that $Y(t, y, \xi)$ is periodic in t; this follows directly from the fact that the trajectory in the phase space is closed (see [3]). Indeed, pushing a little further the analysis of the previous paragraph, we construct $z_\pm$ such that

$$\begin{gathered} |z_+ - z_-| < 2\pi, \quad z_- < z_+, \\ u(z_\pm) = \mathcal{E}, \\ z_- \le y \le z_+, \\ u(z) < \mathcal{E} \quad \forall z \in (z_-, z_+). \end{gathered}$$

Then the particle starting from y with initial speed $-\xi$ reaches either z_+ or z_- in finite time; the speed of the particle is 0 at that moment since

$$|\dot{Y}|^2 = 2(\mathcal{E} - u(Y)),$$

but its acceleration is $-u'(z_\pm) \neq 0$, so the particle turns around and goes back in the reverse direction. It then reaches the other extremity of the interval (z_-, z_+) in finite time, and the same phenomena occurs. Hence after a finite time t_0, the particle is back at its starting point y with the same speed $-\xi$. Consequently, the movement of the particle is periodic in time with period t_0. Thus, we have

$$\mathbb{P}(f)(y,\xi) = \frac{1}{t_0} \int_{t_1}^{t_1+t_0} f(Y(t,y,\xi), \Xi(t,y,\xi))\, dt,$$

where $t_1 \ge 0$ is arbitrary. It is convenient to choose for t_1 the first time when the particle hits z_-. In that case, it is easily seen that t_0 is twice the time it takes to the particle to go from z_- to z_+, so that

$$\frac{t_0}{2} = \int_{t_1}^{t_1+t_0/2} dt = \int_{z_-}^{z_+} \frac{1}{\sqrt{2\,(\mathcal{E} - u(s))}} ds = \left\langle \mathbf{1}_{u<\mathcal{E}} \frac{1}{\sqrt{2\,(\mathcal{E} - u)}} \right\rangle$$

and

$$\begin{aligned} \int_{t_1}^{t_1+\frac{t_0}{2}} f(Y(t,y,\xi), \Xi(t,y,\xi))\, dt &= \left\langle \mathbf{1}_{u<\mathcal{E}} f(s, -\sqrt{2\,(\mathcal{E}-u)}) \frac{1}{\sqrt{2\,(\mathcal{E} - u)}} \right\rangle, \\ \int_{t_1+\frac{t_0}{2}}^{t_1+t_0} f(Y(t,y,\xi), \Xi(t,y,\xi))\, dt &= \left\langle \mathbf{1}_{u<\mathcal{E}} f(s, \sqrt{2\,(\mathcal{E}-u)}) \frac{1}{\sqrt{2\,(\mathcal{E} - u)}} \right\rangle. \end{aligned}$$

Gathering all the terms, we are led to

$$\mathbb{P}(f)(y,\xi) = \frac{\left\langle \mathbf{1}_{u<\mathcal{E}} \left[f\left(\cdot, \sqrt{2\,(\mathcal{E}-u)}\right) + f\left(\cdot, -\sqrt{2\,(\mathcal{E}-u)}\right) \right] \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle}{2 \left\langle \mathbf{1}_{u<\mathcal{E}} \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle} \tag{18}$$

Expressions (17) and (18) are compatible with the ones in [10].

Let us now come back to the case when $N > 1$, and take a function $\varphi(y,\xi) = \varphi_1(y_1,\xi_1) \cdots \varphi_N(y_N,\xi_N)$, where each φ_i is periodic with period 1. We want to compute the limit

$$\frac{1}{T} \int_0^T \varphi_1(Y_1(t,y_1,\xi_1), \Xi_1(t,y_1,\xi_1)) \cdots \varphi_N(Y_N(t,y_N,\xi_N), \Xi_N(t,y_N,\xi_N))\, dt.$$

In general, knowing the behavior of each trajectory (Y_i, Ξ_i) independently is not enough to compute such a product. However, here, we recall that each function $\varphi_i(Y_i(t, y_i, \xi_i), \Xi_i(t, y_i, \xi_i))$ $(1 \le i \le N)$ is periodic in time. The period depends only on $H_i(y_i, \xi_i)$ and on the function u_i. More precisely, setting

$$T_i(\mathcal{E}) := \sqrt{2} \int_0^1 \mathbf{1}_{u_i(z)<\mathcal{E}} \frac{1}{\sqrt{\mathcal{E} - u_i(z)}} \, dz \quad \forall \mathcal{E} > 0, \mathcal{E} \neq u_{\max},$$

$\varphi_i(Y_i(t, y_i, \xi_i), \Xi_i(t, y_i, \xi_i))$ is periodic in time with period $T_i(H_i(y_i, \xi_i))$.

We can thus use the following result :

Lemma 4.1. *Let $f_1, \cdots, f_N \in L^\infty(\mathbb{R})$ such that f_i is periodic with period θ_i, and set $\langle f_i \rangle = \frac{1}{\theta_i} \int_0^{\theta_i} f_i$. Assume that*

$$\frac{k_1}{\theta_1} + \cdots + \frac{k_N}{\theta_N} \neq 0 \quad \forall (k_1, \cdots, k_N) \in \mathbb{Z}^N \setminus \{0\}. \tag{19}$$

Then as $T \to \infty$,

$$\frac{1}{T} \int_0^T f_1(t) \cdots f_N(t) \, dt \to \langle f_1 \rangle \cdots \langle f_N \rangle .$$

Sketch of proof. By density, it is enough to prove the lemma for $f_1, \cdot, f_N \in \mathcal{C}^\infty(\mathbb{R})$. Write f_i as a Fourier series (the series converges thanks to the regularity assumption), and use the fact that for all $\alpha \neq 0$,

$$\frac{1}{T} \int_0^T e^{i\alpha t} \, dt \to 0 \quad \text{as } T \to \infty.$$

□

In the present setting, we deduce the following result :

Proposition 4.1. *Let $\varphi : (y, \xi) \mapsto \varphi_1(y_1, \xi_1) \cdots \varphi_N(y_N, \xi_N)$, where $\varphi_i \in L^\infty_{per}(\mathbb{R}_y \times \mathbb{R}^\xi)$.*

Let $(y, \xi) \in [0, 1)^N \times \mathbb{R}^N$, and let $\theta_i = \theta_i(y_i, \xi_i) = T_i(H_i(y_i, \xi_i))$ for $1 \le i \le N$. Assume that $(\theta_1, \cdots, \theta_N)$ satisfy condition (19). *Then*

$$\mathbb{P}(\varphi)(y, \xi) = \mathbb{P}_1(\varphi_1)(y_1, \xi_1) \cdots \mathbb{P}_N(\varphi_N)(y_N, \xi_N) \tag{20}$$

where each $\mathbb{P}_i$ is the projection in dimension 1 with potential u_i, given by expressions (17) *and* (18).

In particular, when the set

$$\{(y, \xi) \in [0, 1]^N \times \mathbb{R}^N; \ (\theta_1(y_1, \xi_1), \cdots, \theta_N(y_N, \xi_N)) \text{ satisfy condition (19)}\}$$

has zero Lebesgue measure, equality (20) holds almost everywhere. It can then be generalized to arbitrary functions $\varphi \in L^\infty_{\text{per}}(\mathbb{R}^N \times \mathbb{R}^N)$ (always by linearity and density). The correct expression of the projection $\mathbb{P}$ is then

$$\mathbb{P} = \mathbb{P}_1 \circ \mathbb{P}_2 \circ \cdots \circ \mathbb{P}_N, \tag{21}$$

where each projection $\mathbb{P}_i$ acts on the variables (y_i, ξ_i) only. Notice that all projections $\mathbb{P}_i$ thus commute with one another; hence the order in which they are taken is unimportant.

We wish to emphasize that on the open set $\{(y, \xi) \in \mathbb{R}^{2N}, \forall i \in \{1, \cdots, N\} \ H_i(y_i, \xi_i) > \max u_i\}$, the expression (21) is true. Indeed, for $k \in \mathbb{Z}^N \setminus \{0\}$, set

$$A_k := \{t \in \mathbb{R}^N; k_1 t_1 + \cdots + k_N t_N = 0\}.$$

Then A_k is a hyperplane, and we have

$$\begin{aligned} &\left\{(y, \xi), (y, \xi) \in \mathbb{R}^{2N}, H_i(y_i, \xi_i) > \max u_i \ \forall i \text{ and } (\theta_1(y_1, \xi_1), \cdots, \theta_N(y_N, \xi_N)) \text{ satisfy (19)}\right\} \\ = &\bigcup_{k \in \mathbb{Z}^N \setminus \{0\}} \left\{(y, \xi), (y, \xi) \in \mathbb{R}^{2N}, H_i(y_i, \xi_i) > \max u_i \ \forall i \text{ and } (\theta_1(y_1, \xi_1), \cdots, \theta_N(y_N, \xi_N)) \in A_k\right\}. \end{aligned}$$

Hence it suffices to prove that if A is any hyperplane in $\mathbb{R}^N$,

$$\left|\left\{(y,\xi),(y,\xi)\in\mathbb{R}^{2N}, H_i(y_i,\xi_i)>\max u_i\ \forall i \text{ and } (\theta_1(y_1,\xi_1),\cdots,\theta_N(y_N,\xi_N))\in A\right\}\right|=0.$$

Without any loss of generality, assume that

$$A=\{t\in\mathbb{R}^N,\ a_1t_1+\cdots+a_Nt_N=0\},\quad \text{with } a_1\neq 0.$$

Then, using the fact that T_1 is strictly nonincreasing on $(\max u_i,+\infty)$, we can find a open set $C\subset Y\times\mathbb{R}^{N-1}$ and a continuous function $\chi: C\to\mathbb{R}$ such that

$$\begin{aligned} &\left\{(y,\xi),(y,\xi)\in\mathbb{R}^{2N}, H_i(y_i,\xi_i)>\max u_i\ \forall i \text{ and } (\theta_1(y_1,\xi_1),\cdots,\theta_N(y_N,\xi_N))\in A\right\}\\ =\ &\{(y,\chi(y,\xi'),\xi'),\quad (y,\xi')\in C\}\ \cup\ \{(y,-\chi(y,\xi'),\xi'),\quad (y,\xi')\in C\}. \end{aligned}$$

Above, the set $C\subset Y\times\mathbb{R}^{N-1}$ is defined by

$$C:=\left\{(y,\xi')\in Y\times\mathbb{R}^{N-1}, H_i(y_i,\xi_i)>\max u_i,\ i\geq 2 \text{ and } \frac{-1}{a_1}\left(a_2\theta_2(y_2,\xi_2)+\cdots+a_N\theta_N(y_N,\xi_n)\right)>0\right\},$$

and the function χ is defined on C by

$$\chi(y_1,\cdots,y_N,\xi_2,\cdots,\xi_N)=\sqrt{2\left[T_1^{-1}\left(\frac{-1}{a_1}\left(a_2\theta_2(y_2,\xi_2)+\cdots+a_N\theta_N(y_N,\xi_n)\right)\right)-u_1(y_1)\right]},$$

where $T_1^{-1}:(0,\infty)\to(\max u_i,+\infty)$ is the inverse function of T_1.

Since χ is a continuous function, the set

$$\{(y,\chi(y,\xi'),\xi'),\quad (y,\xi')\in C\}$$

has zero Lebesgue measure in $\mathbb{R}^{2N}$ (it is the graph of a continuous curve).

As a consequence, the set

$$\{(y,\xi)\in\mathbb{R}^{2N}, H_i(y_i,\xi_i)>\max u_i\ \forall i \text{ and } (\theta_1(y_1,\xi_1),\cdots,\theta_N(y_N,\xi_N)) \text{ satisfy condition (19)}\}$$

has zero Lebesgue measure.

However, let us mention here that in general, condition (19) cannot be relaxed : indeed, assume for instance that $u_i=u_j:=u$ for $i\neq j$ and assume that the function u is such that

$$\exists y_0>0,\quad u(y)=y^2 \text{ for } |y|<y_0,$$

and $u(y)>y_0^2$ if $y\in[-\frac{1}{2},\frac{1}{2}]\setminus[-y_0,y_0]$.

Then if $|\mathcal{E}|\leq\sqrt{y_0}$, we have

$$T(\mathcal{E})=\int_{-\sqrt{\mathcal{E}}}^{\sqrt{\mathcal{E}}}\frac{1}{\sqrt{\mathcal{E}-y^2}}\,dy=2\int_0^1\frac{1}{\sqrt{1-z^2}}\,dz=:T_0$$

Thus, if $H_i(y_i,\xi_i)\leq\sqrt{y_0}$, then $(Y_i,\Xi_i)(t,y_i,\xi_i)$ is periodic with period T_0. Notice that T_0 does not depend on the energy $H_i(y_i,\xi_i)$

In that case, the function $\varphi(Y(t),\Xi(t))$ is also periodic with period T_0. Thus we have to compute the limit of

$$\frac{1}{T}\int_0^T f_1(t)\cdots f_N(t)\,dt$$

as $T\to\infty$, where the f_i are arbitrary functions with period T_0. It is then easily proved that

$$\frac{1}{T}\int_0^T f_1(t)\cdots f_N(t)\,dt\to\sum_{\substack{k\in\mathbb{Z}^N,\\ k_1+\cdots+k_N=0}} a_{1,k_1}\cdots a_{N,k_N}\tag{22}$$

where

$$a_{j,l} = \frac{1}{T_0} \int_0^{T_0} f_j(t) e^{-\frac{2il\pi t}{T_0}}\, dt, \quad 1 \le j \le N,\ l \in \mathbb{Z}.$$

In general, the right-hand side of (22) differs from $a_{1,0} \cdots a_{N,0}$, and thus

$$\mathbb{P} \neq \mathbb{P}_1 \circ \cdots \circ \mathbb{P}_N$$

for (y, ξ) in a neighbourhood of the origin.

In this regard, let us mention that we believe that there is a slight misprint in [10] concerning the expression of the projection $\mathbb{P}$ when $N = 2$ and for low energies. Indeed, when $\mathcal{E}_1 := H_1(y_1, \xi_1) < \max u_1$ and $\mathcal{E}_2 := H(y_2, \xi_2) < u_2$, it is stated in [10] that

$$\begin{aligned}\mathbb{P}(f)(y,\xi) = \frac{1}{2} \int \quad & \Big\{ f(z_1, z_2, \mathrm{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, \mathrm{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \\ & + f(z_1, z_2, -\mathrm{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, -\mathrm{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \Big\}\, d\nu(z_1, z_2).\end{aligned}$$

where

$$d\nu(z_1, z_2) = \frac{1}{C} \mathbf{1}_{u_1(z_1) < \mathcal{E}_1} \frac{1}{\sqrt{\mathcal{E}_1 - u_1(z_1)}} \mathbf{1}_{u_2(z_2) < \mathcal{E}_2} \frac{1}{\sqrt{\mathcal{E}_2 - u_i(z_2)}}\, dz_1 dz_2,$$

and the constant C is such that ν is a probability measure on $[0,1]^2$.

When $\theta_1(y_1, \xi_1),\ \theta_2(y_2, \xi_2)$ satisfy (19), then $\mathbb{P}(f)(y,\xi) = \mathbb{P}_1 \circ \mathbb{P}_2(f)(y,\xi)$, and thus in that case, the correct expression is rather

$$\begin{aligned}\mathbb{P}(f)(y,\xi) = \frac{1}{4} \int \quad & \Big\{ f(z_1, z_2, \mathrm{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, \mathrm{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \\ & + f(z_1, z_2, \mathrm{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, -\mathrm{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \\ & + f(z_1, z_2, -\mathrm{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, \mathrm{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \\ & + f(z_1, z_2, -\mathrm{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, -\mathrm{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \Big\}\, d\nu(z_1, z_2).\end{aligned}$$

Let us give an explicit example where $\theta_1(y_1, \xi_1), \theta_2(y_2, \xi_2)$ satisfy (19) and $\mathcal{E}_1 < \max u_1$, $\mathcal{E}_2 < \max u_2$. Assume that for $i = 1, 2$, there exists a_i, y_i^0 such that

$$u_i(y_i) = a_i\, |y_i|^2 \quad \forall |y_i| < y_i^0, \quad \text{and} \quad \left(\frac{a_1}{a_2}\right)^2 \notin \mathbb{Q},$$

and $u_i(y_i) > a_i\, |y_i^0|^2$ if $y_i^0 < |y_i| < 1/2$. Indeed, in that case, $T_i(\mathcal{E}_i) = T_0/\sqrt{a_i}$ if $\mathcal{E}_i < a_i\, |y_i^0|^2$, and the condition $a_1^2/a_2^2 \notin \mathbb{Q}$ ensures that $\theta_1(y_1, \xi_1), \theta_2(y_2, \xi_2)$ satisfy (19) for y, ξ in a neighbourhood of the origin.

Thus the expression of [10] is false in that case. In the general case, it is unclear whether a general expression of this kind can be given, considering the discussion around the hypothesis (19) above. Nonetheless, we emphasize that this mistake is of no consequence on the rest of the article [10], and that all the other expressions are compatible with ours.

4.2 Stationary ergodic setting

In the stationary ergodic setting, some of the expressions or properties above are no longer true. The most significant difference occurs when the energy $H(y, \xi) < u_{\max}$; indeed, in that case the particle is not necessarily trapped, depending on the profile of the potential u. Hence, in the rest of the subsection, we focus on the case $H(y, \xi) > u_{\max}$. In that case, the movement of the particle is unbounded and has many similarities with the periodic case. In particular, the particle sees "all the potential" during its evolution, and this will be fundamental in the use of the ergodic theorem.

4.2.1 Expression of $\xi^\sharp(y,\xi,\omega)$

This paragraph is devoted to the proof of proposition 1.3 in the stationary ergodic setting.

We wish to point out that the expressions in the periodic and in the stationary ergodic case when $H(y,\xi,\omega) > u_{\max}$ are exactly the same (compare proposition 1.3 and the end of paragraph 4.1.1). This expression, and more precisely, the equality $\xi^\sharp = \bar{H}'(P)$ for some P, is in fact strongly linked to Aubry-Mather theory. Indeed,

$$\xi^\sharp(y,\xi,\omega) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Xi(t,y,\xi,\omega)\,dt = -\lim_{T\to\infty}\frac{Y(T,y,\xi,\omega)-y}{T},$$

and $\xi^\sharp(y,\xi,\omega)$ is thus (up to a multiplication by -1) the rotation number associated to the Hamiltonian flow starting at (y,ξ). The interested reader should compare our proposition 1.3 to lemma 2.8 in [8] or theorem 4.1 in [9], and our proof to the ones in these articles, and also to the proofs in [15]. We refer to [8, 9, 15] for further references to Aubry-Mather theory and its applications to partial differential equations.

Proof of proposition 1.3. In all the proof, we fix y,ξ,ω such that $H(y,\xi,\omega) > u_{\max}$, and we set $P = P(y,\xi,\omega)$.

The proof is in two steps : first, we prove that

$$\mathbb{P}(L)(y,\xi,\omega) \geq \bar{L}\left(\xi^\sharp(y,\xi,\omega)\right), \tag{23}$$

which is equivalent to

$$\mathbb{P}(L)(y,\xi,\omega) \geq Q\xi^\sharp(y,\xi,\omega) - \bar{H}(Q) \quad \forall Q\in\mathbb{R},$$

and then we exhibit a particular $Q\in\mathbb{R}$ such that equality holds in the previous inequality.

The proof of (23) relies on the following definition of the homogenized Lagrangian (see [18])

$$\forall q\in\mathbb{R},\ \bar{L}(q) = \lim_{T\to\infty}\frac{1}{T}\inf\left\{\int_0^T L(\gamma(s),-\dot\gamma(s),\omega)\,ds,\ \gamma\in W^{1,\infty}((0,T)\times\mathbb{R}^N),\ \gamma(0)=0,\ \gamma(T)=Tq\right\}.$$

The clue of inequality (23) lies in the following remark : since $Y(T)/T \to -\xi^\sharp$ as $T\to\infty$ and L, we could "almost" choose $\gamma = Y$ in the above definition in order to obtain an upper-bound on $\bar{L}(-\xi^\sharp) = \bar{L}(\xi^\sharp)$. Thus we define a function γ which coincides with Y on a large part of the interval $(0,T)$.

Let $T>0$ arbitrary, and let $\lambda\in(0,1)$ be fixed. Define γ by

$$\gamma(s) = Y(s,y,\xi,\omega) \quad \text{for } 1\leq s\leq \lambda T,$$
$$\gamma(0)=0, \quad \gamma(T) = -T\xi^\sharp(y,\xi,\omega),$$
$$\gamma \text{ affine between } 0 \text{ and } 1 \text{ and between } \lambda T \text{ and } T.$$

Then as $T\to\infty$,

$$\frac{1}{T}\int_0^1 L(\gamma(s),-\dot\gamma(s),\omega)\,ds \to 0,$$

and

$$\frac{1}{T}\int_1^{\lambda T} L(\gamma(s),-\dot\gamma(s),\omega)\,ds \to \lambda\mathbb{P}(L)(y,\xi,\omega).$$

There remains to evaluate the contribution of the interval $(\lambda T, T)$. On this interval,

$$\begin{aligned}\dot\gamma(s) &= \frac{1}{T-\lambda T}\left(-T\xi^\sharp(y,\xi,\omega) - Y(\lambda T)\right)\\ &= -\frac{1}{1-\lambda}\xi^\sharp(y,\xi,\omega) - \frac{Y(\lambda T)}{\lambda T}\,\frac{\lambda}{1-\lambda}.\end{aligned}$$

Moreover, for all $(y',\xi',\omega')\in\mathbb{R}^N\times\mathbb{R}^N\times\Omega$,

$$L(y',\xi',\omega') = \frac{1}{2}|\xi'|^2 - u(y',\omega) \leq \frac{1}{2}|\xi'|^2.$$

Thus

$$\int_{\lambda T}^{T} L(\gamma(s), -\dot\gamma(s), \omega)\, ds \le \frac{T-\lambda T}{2} \left| -\frac{1}{1-\lambda}\xi^\sharp(y,\xi,\omega) - \frac{Y(\lambda T)}{\lambda T}\, \frac{\lambda}{1-\lambda}\right|^2.$$

We now pass to the limit as $T\to\infty$, with $\lambda\in(0,1)$ fixed; recall that as $\tau\to\infty$,

$$\frac{Y(\tau,y,\xi,\omega)}{\tau} = -\frac{1}{\tau}\left(\int_0^\tau \Xi(t,y,\xi,\omega)\,dt - y\right) \to -\xi^\sharp(y,\xi,\omega).$$

Thus, for all $\lambda\in(0,1)$,

$$\begin{aligned}\bar L\left(\xi^\sharp(y,\xi,\omega)\right) &\le \lambda \mathbb{P}(L)(y,\xi,\omega) + \frac{1-\lambda}{2}\left| -\frac{1}{1-\lambda}\xi^\sharp(y,\xi,\omega) + \frac{\lambda}{1-\lambda}\xi^\sharp(y,\xi,\omega)\ \right|^2 \\ &\le \lambda \mathbb{P}(L)(y,\xi,\omega) + (1-\lambda)\frac{|\xi^\sharp(y,\xi,\omega)|^2}{2}.\end{aligned}$$

Now, letting $\lambda\to 1$, we obtain inequality (23).

In order to prove the proposition, we have to find a special $Q_0\in\mathbb{R}$ such that

$$\mathbb{P}(L)(y,\xi,\omega) = Q_0\xi^\sharp(y,\xi,\omega) - \bar H(Q_0).$$

This will entail that

$$\mathbb{P}(L)(y,\xi,\omega) = \sup_{Q\in\mathbb{R}}\left(Q\xi^\sharp(y,\xi,\omega) - \bar H(Q)\right) = \bar L\left(\xi^\sharp(y,\xi,\omega)\right),$$

and the sup is obtained for $\xi^\sharp(y,\xi,\omega) = \bar H'(Q_0)$.

The proof of the equality relies on the use of the cell equation for $Q = P = P(y,\xi,\omega)$. Indeed, notice that

$$v(y,\omega) := \operatorname{sgn}(P)\int_0^y \sqrt{2(\bar H(P) - u(z,\omega))}\,dz - Py$$

is a viscosity solution of

$$H(y, P+\nabla_y v, \omega) = \bar H(P),$$

and as $y\to\infty$

$$\frac{1}{y}\int_0^y \sqrt{2(\bar H(P) - u(z,\omega))}\,dz \to E\left[\sqrt{2(\bar H(P) - U)}\right],$$

where $U(\omega) := u(0,\omega)$ for all $\omega\in\Omega$.

By definition of $\bar H$,

$$E\left[\sqrt{2(\bar H(P) - U)}\right] = |P|;$$

consequently,

$$\frac{1}{1+|y|}\left(\operatorname{sgn}(P)\int_0^y \sqrt{2(\bar H(P) - u(z,\omega))}\,dz - Py\right) \to 0 \tag{24}$$

as $y\to\infty$, a.s. in ω. Thus v is a corrector, and $v\in L^\infty(\Omega;\mathcal{C}^1(\mathbb{R}^N))$. Thus the method of characteristics, for instance, can be used to prove that for any couple $(y',\xi') = (y', P+\nabla_y v(y',\omega))$, we have

$$v(y',\omega) = v(Y(T,y',\xi',\omega)) + \int_0^T L(Y(t,y',\xi',\omega), \Xi(t,y',\xi',\omega),\omega)\,dt + P\left[Y(T,y',\xi',\omega) - y'\right] + \bar H(P)T. \tag{25}$$

Before passing to the limit in the above equality, let us prove that we can take $(y',\xi') = (y,\xi)$. First, notice that

$$\nabla_y v(y',\omega) = \operatorname{sgn}(P)\sqrt{2(\bar H(P) - u(y',\omega))} - P,$$

and thus $\operatorname{sgn}\left(P+\nabla_y v(y',\omega)\right) = \operatorname{sgn}(\xi') = \operatorname{sgn}(P) = \operatorname{sgn}(\xi)$. Hence, take $y'=y$. Then $|\xi|^2 = |\xi'|^2$ because $H(y,\xi) = \bar H(P) = H(y,\xi')$ by definition of ξ'. Thus $\xi=\xi'$, and we can take $(y',\xi') = (y,\xi)$ in (25).

Now, we multiply (25) by $1/T$, and pass to the limit as $T \to \infty$. Since

$$\frac{1}{2}|\dot{Y}(t,y,\xi,\omega)| = H(y,\xi,\omega) - u \geq H(y,\xi,\omega) - u_{\max} > 0 \quad \forall t > 0,$$

there exist constants $\alpha, \beta > 0$ depending only on $H(y,\xi,\omega)$ and $u_{\max}$, such that

$$0 < \alpha \leq \left| \frac{Y(T,y,\xi,\omega) - y}{T} \right| \leq \beta \quad \forall T > 0.$$

Consequently, $Y(T) \to \infty$ as $T \to \infty$ and

$$\frac{v(y) - v(Y(T,y,\xi,\omega))}{T} = \frac{v(y) - v(Y(T,y,\xi,\omega))}{Y(T,y,\xi,\omega) - y} \, \frac{Y(T,y,\xi,\omega) - y}{T} \to 0 \quad \text{as } T \to \infty.$$

(remember (24)).

Hence, in the limit we infer

$$\mathbb{P}(L)(y,\xi,\omega) = P\xi^\sharp(y,\xi,\omega) - \bar{H}(P),$$

and the proposition follows.

□

Remark 4.1. *Notice that the proof of inequality* (23) *does not use the fact that the system is integrable, or that $H(y,\xi,\omega) > u_{max}$. Thus* (23) *remains true for small energies, or when the system* (Y,Ξ) *is not integrable.*

4.2.2 Expression of the projection $\mathbb{P}$

The same method as in the periodic case can be used in order to find the expression of the projection $\mathbb{P}$ when $H(y,\xi,\omega) =: \mathcal{E} > u_{\max}$; indeed, in that case, remember that

$$\mathbb{P}(f)(y,\xi,\omega) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega), \omega\right) \, dt$$

and we can use the change of variables

$$dt = \frac{1}{\dot{Y}} \, dY = \frac{1}{-\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(Y,\omega))}} \, dY$$

in order to obtain

$$\begin{aligned} &\int_0^T f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega), \omega\right) \, dt \\ = &\int_y^{Y(T,y,\xi,\omega)} f\left(z, \text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(z,\omega))}, \omega\right) \frac{1}{-\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(z,\omega))}} \, dz. \end{aligned}$$

Since the group transformation (τ_x) is ergodic, and $Y(T) \to \infty$ as $T \to \infty$, for all $\mathcal{E} > u_{\max}$,

$$\begin{aligned} &\frac{1}{Y(T) - y} \int_y^{Y(T,y,\xi,\omega)} f\left(z, \text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(z,\omega))}, \omega\right) \frac{1}{-\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(z,\omega))}} \, dz \to \\ &\qquad \to E\left[F\left(\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(0,\omega))}, \omega\right) \frac{1}{-\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(0,\omega))}} \right] \end{aligned}$$

Thus, we obtain

$$\mathbb{P}(f)(y,\xi,\omega) = \xi^\sharp(y,\xi,\omega)\bar{f}(\text{sgn}(\xi), H(y,\xi,\omega)),$$

where

$$\bar{f}(\eta,\mathcal{E}) = E\left[F\left(\eta\sqrt{2(\mathcal{E} - u(0,\omega))}, \omega\right) \frac{1}{\eta\sqrt{2(\mathcal{E} - u(0,\omega))}} \right].$$

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