Existence of solutions of the hyperbolic Keller-Segel model

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Abstract

We are concerned with the hyperbolic Keller-Segel model with quorum sensing, a model describing the collective cell movement due to chemical signalling with a flux limitation for high cell densities.

This is a first order quasilinear equation, its flux depends on space and time via the solution to an elliptic PDE in which the right hand side is the solution to the hyperbolic equation. This model lacks strong compactness or contraction properties. Our purpose is to prove the existence of an entropy solution obtained, as usual, in passing to the limit in a sequence of solutions to the parabolic approximation.

The method consists in the derivation of a kinetic formulation for the weak limit. The specific structure of the limiting kinetic equation allows for a 'rigidity theorem' which identifies some property of the solution (which might be non-unique) to this kinetic equation. This is enough to deduce a posteriori the strong convergence of a subsequence.

Key-words: Keller-Segel system. Kinetic formulation. Compactness. Entropy inequalities.
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1 Introduction

We consider the hyperbolic Keller-Segel model

$$\begin{cases} \partial_t u + \operatorname{div}\left(\nabla S(t,y)\; g(u)\right) = 0, \quad t > 0,\; y \in \Omega, \\ u(t=0) = u_0 \in L^1 \cap L^\infty(\Omega), \quad 0 \le u_0 \le 1 \text{ a.e}, \\ -\Delta S + S = u \quad \text{in } \Omega, \\ \nabla S \cdot n_\Omega = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, the function $g(u)$ is given by

$$g(u) = u(1-u)$$

therefore we restrict ourselves to solutions satisfying $0 \le u(t,x) \le 1$. The problem is posed on Ω, it is any bounded domain in $\mathbb{R}^N$, with $\mathcal{C}^1$ boundary, and $n_\Omega(y)$ is the outward normal to Ω at $y \in \partial\Omega$. One can also take the torus $\Omega = \Pi_{i=1}^N (0, T_i)$, with $T_i > 0$, and with periodic boundary conditions in Ω for S; the results and proofs are the same. Notice in particular that the normal flux in the equation on u vanishes on $\partial\Omega$ and thus the boundary is characteristic; therefore we do not need boundary conditions for u (this prevents us from investigating questions which rise specific difficulties, see [19] for instance).

This model represents the density $u(t,y)$ of cells moving with a collective chemotactic attraction through the chemical potential S. Their sensitivity is limited by the so-called 'quorum sensing' term $(1-u)$ in $g(u)$. It enters a general class of problems in the description of cells movement ([17, 23, 13, 14, 11, 12, 22, 5]). Usually a diffusion term is added to represent the random motion of the cells and the above model corresponds to the small viscosity limit which has been advocated by several authors, see [18, 25, 10, 4] for these aspects.

This derivation implies that the system (1) comes with an entropy structure as usual ([24, 6]). But extra terms enter in this entropy structure because of the space dependency of the flux and this leads to a specific difficulties (see [3, 1, 2, 8] and the references therein). For any $\mathcal{C}^2$ convex function η (the so-called entropy), we have

$$\frac{\partial}{\partial t}\eta(u) + \operatorname{div}(\nabla S\, q(u)) + (u - S)\left[q - g\eta'\right](u) \le 0, \tag{2}$$

where $q'(\xi) := g'(\xi)\eta'(\xi)$ for $\xi \in \mathbb{R}$. This accounts for the correct jump condition on possible discontinuities of u. But due to the dependency of the flux on $\nabla S(t, y)$, the above model poses specific difficulties compared to the usual theory of quasilinear scalar conservation laws: no a priori compactness is known in dimension larger than 1 (no BV bounds or L^1 compactness), contraction principle or uniqueness are not known and averaging lemma for the kinetic formulation (see below) do not apply (because the transport is mostly one dimensional in the direction ∇S). Even time continuity in L^1 does not follow from the method we develop in this paper. As a consequence we do not know if the full family of solutions to the diffusion approximation converges, but only subsequences. All these questions are left open and seem difficult.

Consequently, our proof relies on the weak limit of the diffusion approximation of (1) that we study through its kinetic formulation. Passing to the limit we obtain a weak form of the kinetic formulation of the hyperbolic limit. The main ingredient then is to prove a *rigidity theorem* for the solution which implies that the weak limit is a usual entropy solution and that subsequences converge strongly. The kinetic formulation and the main results are presented in the next subsection. The diffusion limit is studied in section 3, and the rigidity theorem is proved in section 4. Finally, we analyze the long time behavior of solutions in Section 5. Some technical aspects are left in an appendix.

2 Main results

Our main existence result is the following:

Theorem 2.1. *The system* (1) *has a solution* $u \in L^\infty(\mathbb{R}^+ \times \Omega)$, $S \in L^\infty(\mathbb{R}^+; W^{2,q}(\Omega))$ *for* $1 \le q < \infty$, *satisfying* $0 \le u(t, y) \le 1$, $0 \le S(t, y) \le 1$, *and all the entropy inequalities* (2) *(in the weak sense, with initial data* $\eta(u_0)$*).*

Because our method is based on weak limits as mentioned earlier, it is more convenient to use the kinetic formulation of (1) (see [15, 16, 21, 7] for the theory of kinetic formulations and recent applications). It is a way to represent all the inequalities (2) in a single equation on the unknown defined on $[0,\infty)\times\Omega\times\mathbb{R}$, $f(t,y,\xi)=\mathbf{1}_{\xi<u(t,y)}$, namely

$$\begin{cases} \frac{\partial f}{\partial t}+(\xi-S)g(\xi)\frac{\partial f}{\partial \xi}+g'(\xi)\nabla_y S\cdot\nabla_y f=\frac{\partial m}{\partial \xi}, \\ m(t,y,\xi)\geq 0 \quad \text{a bounded measure on } [0,T]\times\Omega\times\mathbb{R}, \quad \forall T>0, \\ f(0,y,\xi)=\mathbf{1}_{\xi<u_0(y)}, \\ -\Delta S+S=u:=\int\limits_0^\infty f(t,y,\xi)d\xi \quad \text{in } \Omega, \qquad \nabla S\cdot n_\Omega=0 \quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

This is equivalent to (2), and one can recover (2) from (3) using that $\eta(u)=\int\eta'(\xi)\mathbf{1}_{\xi<u(t,y)}d\xi$ because we can always take $\eta(u)=0$ for $u\leq 0$ without loss of generality; see also Section 3 for an alternative derivation.

The outcome of our proof is the following rigidity theorem:

Theorem 2.2. *Consider a weak solution to the kinetic equation*

$$\begin{cases} \frac{\partial f}{\partial t}+(\xi-S)g(\xi)\frac{\partial f}{\partial \xi}+g'(\xi)\nabla_y S\cdot\nabla_y f+R(t,y,\xi)=\frac{\partial m}{\partial \xi}, \\ m(t,y,\xi)\geq 0 \quad \textit{a bounded measure on } [0,T]\times\Omega\times\mathbb{R}, \quad \forall T>0, \\ f(0,y,\xi)=\mathbf{1}_{\xi<u_0(y)}, \\ -\Delta S+S=u:=\int\limits_0^\infty f(t,y,\xi)d\xi \quad \textit{in } \Omega, \qquad \nabla S\cdot n_\Omega=0 \quad \textit{on } \partial\Omega. \end{cases} \tag{4}$$

which satisfies the properties
(i) $0\leq f(t,y,\xi)\leq 1$ *and* $f=1$ *for* $\xi<0$, $f=0$ *for* $\xi>1$, f *is nonincreasing in* ξ,
(ii) there exists a constant $C>0$ *such that* $|R|\leq Cf(1-f)$ *almost everywhere,*
(iii) the measure m *vanishes for* $\xi<0$ *or* $\xi>1$.
Then, we have $f(t,y,\xi)=\mathbf{1}_{\xi<u(t,y)}$ *and* $u(t,y)$ *is an entropy solution to* (1).

The proof of these two results is given in the next sections. The strategy in the following : as in [10], we take a parabolic approximation of (1), and we intend to pass to the limit in its solution u^ε as the viscosity vanishes. However, unlike in [10] and as mentioned earlier, the problem (1) lacks a priori compactness bounds for u^ε when $N>1$. Hence, we rather pass to the limit in a kinetic formulation of the approximate problem. The weak limit of the sequence $f^\varepsilon = \mathbf{1}_{\xi < u^\varepsilon(t,y)}$ satisfies equation (4), with a remainder R which can be explicitly computed in terms of f and which satisfies (ii). Thus theorem 2.2 implies in turn that u^ε converges strongly to u. Let us finally mention that an alternative proof for the local existence of strong solutions can be carried out, see [4]; however, as stressed by M. Burger, Y. Dolak and C. Schmeiser in [4], their strategy does not yield any information on the global existence of weak solutions.

3 The parabolic limit

In this section, we introduce and study the approximate parabolic system with $\varepsilon > 0$:

$$\begin{cases} \partial_t u^\varepsilon + \mathrm{div}\,(\nabla S^\varepsilon(t,y)\, u^\varepsilon(1-u^\varepsilon)) - \varepsilon \Delta u^\varepsilon = 0, \quad t>0,\ y \in \Omega \\ u^\varepsilon(t=0) = u_0 \in L^1 \cap L^\infty(\Omega), \quad 0 \le u_0 \le 1 \text{ a.e}, \\ -\Delta S^\varepsilon + S^\varepsilon = u^\varepsilon \quad \text{in } \Omega, \\ \nabla S^\varepsilon \cdot n_\Omega = 0 \quad \text{on } \partial\Omega, \\ \nabla u^\varepsilon \cdot n_\Omega = 0 \quad \text{on } \partial\Omega \quad \text{for a.e. } t>0. \end{cases} \tag{5}$$

Our goal is to pass to the weak limit in this system but we first state the following result

Proposition 3.1. *There exists a unique solution $(u^\varepsilon, S^\varepsilon) \in L^2_{loc}(0,\infty; H^1(\Omega)) \times L^\infty(0,\infty; H^1(\Omega))$ of the problem (5) and it satisfies the following bounds : for all $1 \le q < \infty$, for all $T > 0$, there exist constants $C_1(N,\Omega,q)$, $C_2(N,\Omega,T)$ such that*

$$0 \le u^\varepsilon(t,y) \le 1 \quad a.e.\ on\ [0,\infty)\times\Omega, \tag{6}$$

$$0 \le S^\varepsilon(t,y) \le 1 \quad a.e.\ on\ [0,\infty)\times\Omega, \tag{7}$$

$$||S^\varepsilon||_{L^\infty(0,\infty;W^{2,q}(\Omega))} \le C_1, \tag{8}$$

$$\sqrt{\varepsilon}||\nabla u^\varepsilon||_{L^2((0,T)\times\Omega)} + ||\partial_t S^\varepsilon||_{L^2((0,T);H^1(\Omega))} \le C_2. \tag{9}$$

And for any $\mathcal{C}^2$ convex function η, we have with the notation in (2),

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \mathrm{div}(\nabla S^\varepsilon\, q(u^\varepsilon)) + (u^\varepsilon - S^\varepsilon)\left[q - g\eta'\right](u^\varepsilon) - \varepsilon\Delta\eta(u^\varepsilon) \le 0. \tag{10}$$

Proof. Existence and uniqueness of $(u^\varepsilon, S^\varepsilon)$ are easily proved thanks to semi-group techniques. The bounds (6) follows from the maximum principle because 0 and 1 are solutions for all drifts ∇S^ε, the bound (7) also follows from the maximum principle, whereas (8) is the regularizing effect for elliptic equations with smooth coefficients.

The first bound on ∇u^ε in (9) is obtained by multiplying by u^ε the evolution equation on u^ε. Eventually, differentiating the equation giving S^ε with respect to t gives

$$-\Delta\partial_t S^\varepsilon + \partial_t S^\varepsilon = -\text{div}\,(\nabla S^\varepsilon\, u^\varepsilon(1-u^\varepsilon)) + \varepsilon\Delta u^\varepsilon,$$

and the right-hand side is bounded in $L^2_{\text{loc}}(0,\infty; H^{-1}(\Omega))$ uniformly in ε; the second bound of (9) follows.

The entropy inequality (10) is obtained by multiplication of the evolution equation by $\eta'(u^\varepsilon)$ and using the chain rule. □

Next, we pass to the limit in the system (5). However, because they do not provide strong compactness, the bounds on the sequence u^ε are insufficient to pass to the limit in the nonlinear term

$$\nabla S^\varepsilon u^\varepsilon(1-u^\varepsilon).$$

In [10], for $N=1$, strong compactness is obtained thanks to uniform BV bounds on the sequence u^ε; however, as we have already pointed out, such bounds no longer hold when $N>1$. Consequently, we pass to the (weak) limit in the kinetic formulation for problem (5). Our next goal is to present this argument.

We wish to take $\eta(u)=(u-\xi)_+$ in (10), with $\xi\in\mathbb{R}$. However, such an entropy is not $\mathcal{C}^2$; thus, for $\delta>0$, we consider a $\mathcal{C}^2$, convex function φ_δ such that

$$\varphi_\delta(u) \underset{\delta\to 0}{\longrightarrow} u_+ \quad \text{in } L^\infty_{\text{loc}}(\mathbb{R}).$$

The entropy $\varphi_\delta(u-\xi)$ satisfies inequality (10), and $\varphi_\delta(u^\varepsilon-\xi)$ converges strongly in $L^\infty((0,\infty)\times\Omega)$ towards $(u^\varepsilon-\xi)_+$. Thus, passing to the limit as $\delta\to 0$, we infer that (10) is satisfied with $\eta(u)=(u-\xi)_+$.

Then, we differentiate (in the distributional sense) the inequality obtained with respect to ξ. This yields

$$\frac{\partial f^\varepsilon}{\partial t} + (\xi - S^\varepsilon)g(\xi)\frac{\partial f^\varepsilon}{\partial \xi} + g'(\xi)\nabla_y S^\varepsilon\cdot\nabla_y f^\varepsilon - \varepsilon\Delta_y f^\varepsilon = \frac{\partial m^\varepsilon}{\partial\xi}, \tag{11}$$

where $m^\varepsilon(t,y,\xi)$ is a nonnegative measure on $[0,\infty)\times\Omega\times\mathbb{R}$. It can be written explicitely in terms of u^ε, namely

$$\begin{aligned} m^\varepsilon(t,y,\xi) \;&:=\; -\{\partial_t(u^\varepsilon-\xi)_+ + \text{div}\,(\nabla S^\varepsilon\,\mathbf{1}_{\xi<u^\varepsilon}(g(u^\varepsilon)-g(\xi))) \\ &\qquad - (u^\varepsilon - S^\varepsilon)\mathbf{1}_{\xi<u^\varepsilon} g(u^\varepsilon) - \varepsilon\Delta_y(u^\varepsilon-\xi)_+\} \\ &=\; \varepsilon|\nabla_y u^\varepsilon(t,y)|^2\delta(\xi = u^\varepsilon(t,y)). \end{aligned}$$

Notice that $0 \leq f^\varepsilon \leq 1$ almost everywhere, and $f^\varepsilon(t,y,\xi) = 0$ when $\xi > 1$, $f^\varepsilon(t,y,\xi) = 1$ when $\xi < 0$. Moreover, $m^\varepsilon(t,y,\xi) = 0$ when $\xi < 0$ or $\xi > 1$ (in the sense of distributions), and $\{m^\varepsilon(t,y,\xi)\}_{\varepsilon>0}$ is a family of bounded measures on $[0,T] \times \Omega \times \mathbb{R}$, $\forall T > 0$.

Hence, there exists a subsequence, still denoted by ε, and functions $u = u(t,y) \in L^\infty((0,\infty) \times \Omega)$, $f = f(t,y,\xi) \in L^\infty((0,\infty) \times \Omega \times \mathbb{R})$, $S = S(t,y) \in L^\infty(0,\infty; W^{2,q}(\Omega))$, and a nonnegative measure $m = m(t,y,\xi)$ such that, locally in time,

$$\begin{gathered} u^\varepsilon \rightharpoonup u \quad w^* - L^\infty, \\ f^\varepsilon \rightharpoonup f \quad w^* - L^\infty, \\ m^\varepsilon \rightharpoonup m \quad w - M^1, \\ S^\varepsilon \to S \quad \text{in } L^p_{\text{loc}}(0,\infty; W^{1,p}(\Omega)) \end{gathered}$$

for all p, $1 \leq p < \infty$.

Thus, we can pass to the limit as $\varepsilon \to 0$ in equation (11). All the terms can pass to the limit because they are written as weak-strong products except $g'(\xi)\nabla_y S^\varepsilon \cdot \nabla_y f^\varepsilon$ which yields an extra term. Indeed, we can write

$$\begin{aligned} g'(\xi)\nabla_y S^\varepsilon \cdot \nabla_y f^\varepsilon &= \operatorname{div}_y \left(g'(\xi)\nabla_y S^\varepsilon\, f^\varepsilon\right) - g'(\xi)\Delta_y S^\varepsilon\, f^\varepsilon \\ &= \operatorname{div}_y \left(g'(\xi)\nabla_y S^\varepsilon\, f^\varepsilon\right) + (u^\varepsilon - S^\varepsilon) g'(\xi) f^\varepsilon. \end{aligned}$$

In the sense of distributions, as $\varepsilon \to 0$, we have

$$\begin{gathered} \operatorname{div}_y \left(g'(\xi)\nabla_y S^\varepsilon\, f^\varepsilon\right) \rightharpoonup \operatorname{div}_y \left(g'(\xi)\nabla_y S\, f\right), \\ S^\varepsilon\, g'(\xi) f^\varepsilon \rightharpoonup S\, g'(\xi)\, f. \end{gathered}$$

But at this stage, we cannot assert that the weak limit of $u^\varepsilon f^\varepsilon$ is uf (but it is possible to identify it, see (15) below). Nevertheless, we know that $\{u^\varepsilon f^\varepsilon\}_{\varepsilon>0}$ is bounded in L^∞; thus, extracting a further subsequence if necessary, there exists a function $\rho = \rho(t,y,\xi)$ such that

$$u^\varepsilon\, f^\varepsilon \rightharpoonup \rho \quad w^* - \; L^\infty. \tag{12}$$

Consequently,

$$g'(\xi)\nabla_y S^\varepsilon \cdot \nabla_y f^\varepsilon \rightharpoonup g'(\xi)\nabla_y S \cdot \nabla_y f + g'(\xi)\left(\rho - u\, f\right),$$

and f is a solution of

$$\begin{gathered} \partial_t f + (\xi - S) g(\xi) \partial_\xi f + g'(\xi)\nabla_y S \cdot \nabla_y f + g'(\xi)(\rho - uf) = \partial_\xi m, \\ -\Delta S + S = u(t,y) \quad \text{in } \Omega, \\ \nabla_y S \cdot \nabla n_\Omega = 0 \quad \text{on } \partial\Omega, \\ f(t=0,y,\xi) = \mathbf{1}_{\xi < u_0(y)}. \end{gathered} \tag{13}$$

Moreover, f, u and m inherit the following properties

$$
\begin{gathered}
0 \leq f \leq 1 \quad \text{a.e.}, \\
f(t,y,\xi) = 0 \quad \text{when } \xi > 1, \quad f(t,y,\xi) = 1 \quad \text{when } \xi < 0, \\
m(t,x,\xi) = 0 \quad \text{when } \xi > 1 \text{ or } \xi < 0, \\
\int_0^T \int_\Omega \int_{\mathbb{R}} m(t,y,\xi)\, dt\, dy\, d\xi < \infty \quad \forall T > 0.
\end{gathered}
$$

And there exists a nonnegative measure $\nu(t,y,\xi)$ such that $\nu((0,T)\times\bar{\Omega}\times\mathbb{R}) < \infty$ for all $T > 0$ and

$$
\partial_\xi f(t,x,\xi) = -\nu(t,x,\xi) \leq 0 \tag{14}
$$

in the sense of distributions. This follows from the fact that

$$
\partial_\xi f^\varepsilon(t,y,\xi) = -\delta(\xi - u^\varepsilon(t,y)).
$$

This means that we have derived the properties (i) and (iii) assumed in Theorem 2.2. The remainder term R is here equal to $R(t,y,\xi) := g'(\xi)(\rho - uf)(t,y,\xi)$. There remains to derive a formula for ρ which we do now.

Let $\varphi_1 \in \mathcal{C}_0^\infty([0,\infty)\times\Omega)$, $\varphi_2 \in \mathcal{C}_0^\infty(\mathbb{R})$ be test functions.

Then

$$
\begin{aligned}
& \int_0^\infty \int_{\Omega\times\mathbb{R}} \rho(t,y,\xi)\varphi_1(t,y)\, \varphi_2'(\xi)\, dt\, dy\, d\xi \\
= \quad & \lim_{\varepsilon\to 0} \int_0^\infty \int_{\Omega\times\mathbb{R}} u^\varepsilon(t,x) f^\varepsilon(t,x,\xi)\varphi_2'(\xi)\varphi_1(t,y)\, dt\, dy\, d\xi \\
= \quad & \lim_{\varepsilon\to 0} \int_0^\infty \int_\Omega u^\varepsilon(t,x)(\varphi_2(u^\varepsilon(t,x)))\varphi_1(t,y)\, dt\, dy \\
= \quad & \lim_{\varepsilon\to 0} \int_0^\infty \int_{\Omega\times\mathbb{R}} \frac{d}{d\xi}(\xi\varphi_2(\xi)) f^\varepsilon(t,x,\xi)\varphi_1(t,y)\, dt\, dy\, d\xi \\
= \quad & \int_0^\infty \int_{\Omega\times\mathbb{R}} \frac{d}{d\xi}(\xi\varphi_2(\xi)) f(t,x,\xi)\varphi_1(t,y)\, dt\, dy\, d\xi \\
= \quad & \int_0^\infty \int_{\Omega\times\mathbb{R}} \frac{d}{d\xi}(\xi\varphi_2(\xi)) f(t,x,\xi)\varphi_1(t,y)\, dt\, dy\, d\xi
\end{aligned}
$$

Consequently,

$$
-\frac{\partial}{\partial\xi}\left[\rho - \xi f\right] = f.
$$

Next, we integrate this equation on $\mathbb{R}$ (t, y are treated as fixed parameters), with the boundary conditions $f(t, y, \xi) = 0$ and $\rho(t, y, \xi) = 0$ when $\xi > 1$. We get

$$\rho(t, y, \xi) - \xi f(t, y, \xi) = \int_{\xi}^{\infty} f(t, y, \xi')\, d\xi'. \tag{15}$$

Let us now prove that property (ii) follows from equation (15):

Lemma 3.1. *For $T > 0$, set*

$$C := \limsup ||u^{\varepsilon}||_{L^{\infty}}((0, T) \times \Omega).$$

(Notice that $C \leq 1$ here). Then, with ρ given in (15)*, we have*

$$|\rho(t, y, \xi) - u(t, y)\, f(t, y, \xi)| \leq C f(t, y, \xi)\, (1 - f(t, y, \xi))$$

for a.e. $t \in (0, T)$, $y \in \Omega$, $\xi \in \mathbb{R}$.

Proof. From (15), for almost every $(t, y, \xi) \in [0, \infty) \times \Omega \times \mathbb{R}$,

$$\begin{aligned}
& \rho(t, y, \xi) - u(t, y) f(t, y, \xi) \\
= \; & \xi f(t, y, \xi) + \int_{\xi}^{\infty} f(t, y, \xi')\, d\xi' - u(t, y) f(t, y, \xi) \\
= \; & \int_0^{\xi} d\xi'\, f(t, y, \xi) + \int_{\xi}^{\infty} f(t, y, \xi')\, d\xi' \\
& - f(t, y, \xi) \int_0^{\xi} f(t, y, \xi')\, d\xi' - f(t, y, \xi) \int_{\xi}^{\infty} f(t, y, \xi')\, d\xi' \\
= \; & \left[\int_0^{\xi} (1 - f(t, y, \xi')) d\xi' \right] f(t, y, \xi) + \left[\int_{\xi}^{\infty} f(t, y, \xi')\, d\xi' \right] (1 - f(t, y, \xi)).
\end{aligned}$$

Now, remember that $f(t, y, \xi)$ is decreasing with respect to ξ (recall (14)). Therefore $f(t, y, \xi') \leq f(t, y, \xi)$ for $\xi' \geq \xi$, and $1 - f(t, y, \xi') \leq 1 - f(t, y, \xi)$ for $\xi' \leq \xi$. And for $t \in [0, T]$, $y \in \Omega$, $f(t, y, \xi) = 0$ for $\xi > \limsup ||u^{\varepsilon}||_{L^{\infty}}((0, T) \times \Omega)$.

Eventually, we obtain

$$0 \leq \rho(t, y, \xi) - u(t, y) f(t, y, \xi) \leq \limsup ||u^{\varepsilon}||_{L^{\infty}} [f(1 - f)]\, (t, y, \xi).$$

$\square$

At this stage we have derived the full kinetic formulation for our problem, which means that the assumptions of Theorem 2.2 have been obtained in the (weak) limit of solutions to the parabolic equation (5). We can turn to its proof.

4 Proof of the rigidity Theorem 2.2

The technique introduced in [20] is then to compare f and f^2 in order to prove that f only takes the values 0 and 1 almost everywhere. Thanks to the monotony assumption in (i) (see (14)), we can then deduce easily that there exists $u = u(t, y)$ such that $f(t, x, \xi) = \mathbf{1}_{\xi < u(t,y)}$.

Hence, we multiply (4) by $2f$ and we formally derive an equation for f^2; the difference $f - f^2$ satisfies

$$\frac{\partial}{\partial t}(f - f^2) + (\xi - S)g(\xi)\partial_\xi(f - f^2) + g'(\xi)\nabla_y S \cdot \nabla_y(f - f^2) + R(1 - 2f) = \\ = \partial_\xi m(1 - 2f). \quad (16)$$

We emphasize that this calculation, and the following, seems entirely formal; indeed, since f is not smooth, the chain rule $2\partial_t f f = \partial_t f^2$ for instance, has to be justified. Thus, regularizations in (t, y, ξ) are necessary in order to make the argument rigorous. Those are fairly standard (see [9, 21, 20]), and will be detailed in the Appendix.

It can be seen in the above equation that the key of our method is the assumption (ii) on the term R. In the case where R is equal to $R = g'(\rho - uf)$, with ρ given by (15), the inequality in assumption (ii) has been derived in lemma 3.1.

Now, we integrate (16) on $\Omega \times \mathbb{R}$ (notice that for $\xi < 0$ or $\xi > 1$, $f = f^2$). We get

$$\begin{aligned} \frac{d}{dt}\int_{\Omega\times\mathbb{R}} (f - f^2) \quad &\le \quad \int_{\Omega\times\mathbb{R}} (f - f^2) \left\{ \frac{\partial}{\partial \xi}\left[(\xi - S)g(\xi)\right] + \Delta_y S g'(\xi) \right\} \\ &\quad + C\int_{\Omega\times\mathbb{R}} |(1 - 2f)|\,(f - f^2) \\ &\quad + 2\int_{\Omega\times\mathbb{R}} m(t, y, \xi)\,\partial_\xi f(t, y, \xi)\,dy\,d\xi \\ &\le \quad C\int_{\Omega\times\mathbb{R}} (f - f^2). \end{aligned} \quad (17)$$

In the above inequality, we have used the facts that $f - f^2 \ge 0$ because $0 \le f \le 1$, and $0 \le S \le 1$ by the maximum principle. Also,

$$|\Delta S| = |S - u| \le 1,$$

$$|g(\xi)|, |g'(\xi)| \le 1 \quad \forall \xi \in [0, 1],$$

$$\int_{\Omega\times\mathbb{R}} m(t, y, \xi)\,\partial_\xi f(t, y, \xi)\,dy\,d\xi = -\int_{\Omega\times\mathbb{R}} m(t, y, \xi)\nu(t, y, \xi)\,dy\,d\xi \le 0.$$

Consequently, by Gronwall's lemma, we get

$$0 \le \int_{\Omega\times\mathbb{R}} (f-f^2)(t,y,\xi)\,dy\,d\xi \le e^{Ct}\int_{\Omega\times\mathbb{R}} (f-f^2)(t=0,y,\xi)\,dy\,d\xi$$

But $f(t=0,y,\xi) = \mathbf{1}_{\xi<u_0(y)}$, and thus $(f-f^2)(t=0)=0$. We deduce that $f(t,y,\xi) = f^2(t,y,\xi)$ for a.e. (t,y,ξ), and $f=0$ or $f=1$ almost everywhere. Since f is decreasing in ξ, $f=\mathbf{1}_{\xi<u(t,y)}$, and it is easily checked that in that case, the remainder term R is equal to zero (recall assumption (ii) in Theorem 2.2). If $R = g'(\rho - uf)$, with ρ given by (15), we deduce that $\rho = uf$. Hence f is a solution of

$$\partial_t f + (\xi - S)g(\xi)\partial_\xi f + \nabla_y S\cdot\nabla_y f g'(\xi) = \partial_\xi m,$$

and u is an entropy solution of

$$\partial_t u + \mathrm{div}_y(\nabla S g(u)) = 0, \quad t>0,\ y\in\Omega.$$

Remark 4.1. *It can be checked that*

$$\mathbf{1}_{\xi<u^\varepsilon(t,y)} \rightharpoonup \mathbf{1}_{\xi<u(t,y)}\ w^*-L^\infty \quad\iff\quad u^\varepsilon\to u \quad in\ L^1_{loc}((0,\infty)\times\Omega).$$

Hence, we have proved here the following more general result : let $\{(u_n,S_n)\}_{n\ge0}$ be a sequence such that $f_n := \mathbf{1}_{\xi<u_n(t,y)}$ satisfies

$$\begin{gathered}\partial_t f_n + (\xi-S_n)g(\xi)\partial_\xi f_n + g'(\xi)\nabla S_n\cdot\nabla f_n = \partial_\xi m_n + r_n,\\ -\Delta S_n + S_n = u_n,\\ 0\le u_n\le 1,\\ f_n(t=0) = 1_{\xi<u_n^0(y)},\end{gathered}$$

with m_n a nonnegative measure and $r_n \rightharpoonup 0$ in the sense of distributions.

Assume that $f_n \rightharpoonup f(t,y,\xi)$ w^*-L^∞ and $u_n^0(y)\to u^0(y)$ strongly in $L^1(\Omega)$ as $n\to\infty$.

Then there exists $u=u(t,y)\in L^\infty((0,\infty)\times\Omega)$ such that $f=\mathbf{1}_{\xi<u(t,y)}$ and $u_n\to u$ in $L^1((0,T)\times\Omega)$ for all $T>0$. And u is an entropy solution of (1).

5 Long-time behavior

We wish to mention here a few simple facts on the long-time behavior of a solution (u,S) of system (1). Our motivation comes from the unusual complexity of the behavior exhibited in [4, 10] for this limit. From their study there appears to be differences between the large time dynamics of the parabolic system (5) and the hyperbolic system (1). The stability of the steady states of (1) is also discussed in these references. The numerical simulations and formal computations presented in these two papers also convey a rather good insight of the long time behavior of solutions. Precisely, the numerical simulations indicate that for the hyperbolic system (1) in dimension one, solutions converge to piecewise constant steady states as time goes to infinity. In these steady states, regions of vacuum ($u_\infty = 0$ below) are separated from regions where cells aggregate ($u_\infty = 1$ below) by entropic shocks. On the contrary, in the parabolic system (5), when the parameter ε is large enough ($\varepsilon > \frac{1}{4}$), solutions converge to the constant solution. And when the diffusivity parameter ε is small, a metastable behavior occurs : solutions first get close to the piecewise constant steady states of the hyperbolic system, and then the regions of cell aggregates (called 'plateaus') move slowly and at last merge with one another.

This section aims at proving that any entropy solution to (1), as built in Theorem 2.1, converges (in a sense detailed in the next proposition) to a steady state solution:

$$\begin{cases} \operatorname{div}\left(\nabla S_\infty \, u_\infty(1-u_\infty)\right) = 0 \quad \text{in } \Omega, \\ -\Delta S_\infty + S_\infty = u_\infty \quad \text{in } \Omega, \\ \nabla S_\infty \cdot n_\Omega = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{18}$$

Our analysis relies on the energy dissipation inherited from the natural free energy structure for the chemotaxis systems ([5, 22, 14])

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(t)S(t) &= \frac{d}{dt}||S(t)||^2_{H^1(\Omega)} \\ &= 2\int_\Omega |\nabla S(t,y)|^2\, u(t,y)(1-u(t,y))\, dy \geq 0, \end{aligned} \tag{19}$$

and consequently.

$$\int_\tau^\infty \int_\Omega |\nabla S(t,y)|^2\, u(t,y)(1-u(t,y))\, dy\, dt \to 0 \quad \text{as } \tau \to \infty. \tag{20}$$

The equality (19) is proved in [10] when $N = 1$, but the dimension does not play a significant part here; we reproduce a short proof for the reader convenience. First, notice that

$$-\Delta S + S = u, \qquad -\Delta \partial_t S + \partial_t S = \partial_t u,$$

and thus, multiplying the first equation by $\partial_t S$ and the second by S, after integration by parts we obtain

$$\begin{aligned}\int_\Omega u\partial_t S &= \int_\Omega (\nabla S\cdot\nabla\partial_t S + S\,\partial_t S)\\ &= \int_\Omega \partial_t u S = \frac{1}{2}\frac{d}{dt}||S||^2_{H^1(\Omega)}.\end{aligned}$$

Notice also that $\int_\Omega uS = ||S||^2_{H^1}$. Consequently,

$$\begin{aligned}\frac{d}{dt}\int_\Omega uS &= 2\int_\Omega S\partial_t u\\ &= -2\int_\Omega S\,\mathrm{div}\,(\nabla S g(u)) = 2\int_\Omega |\nabla S|^2\, g(u).\end{aligned}$$

Therefore, we are led to

$$\int_\Omega u(t,y)\frac{\partial S(t,y)}{\partial t}\,dy = \int_\Omega S(t,y)\frac{\partial u(t,y)}{\partial t}\,dy = \frac{1}{2}\frac{d}{dt}||S||^2_{H^1(\Omega)}.$$

And (19) follows.

Integrating this equality from $t = 0$ to $t = T$, we obtain

$$\int_0^T\int_\Omega |\nabla S|^2 g(u) = \frac{1}{2}\left[\int_\Omega (u(T)S(T) - u(t=0)S(t=0))\right] \leq \frac{1}{2}|\Omega|.$$

Thus

$$\int_0^\infty\int_\Omega |\nabla S|^2 g(u) \leq \frac{1}{2}|\Omega| < +\infty$$

and (20) follows.

We can now state our main result

Proposition 5.1. *Let (u, S) be a global weak solution of* (1) *as in Theorem 2.1. Then, for $k \in \mathbb{N}$, let $u_k(t, y) := u(t + k, y)$, $S_k(t, y) := S(t + k, y)$, $t > 0$, $y \in \Omega$. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such as when $k \to \infty$,*

$$u_{n_k}(t,y) \rightharpoonup u_\infty(y) \quad w^* - L^\infty(\mathbb{R}^+\times\Omega), \tag{21}$$

$$S_{n_k}(t,y) \to S_\infty(y) \quad in\ L^2(0,T;H^1(\Omega)), \qquad \forall T>0, \tag{22}$$

$$\int_0^T\int_\Omega \mathbf{1}_{\nabla S\neq 0}|u_{n_k}(t,y) - u_\infty(y)|\,dt\,dy \to 0, \tag{23}$$

where $u_\infty = u_\infty(y) \in L^\infty(\Omega)$, $S_\infty = S_\infty(y) \in H^2(\Omega)$ are solutions to (18) *and $0 \leq u_\infty(y) \leq 1$, $0 \leq S_\infty(y) \leq 1$ and $|\nabla S_\infty|u_\infty(1 - u_\infty) = 0$.*

Proof. First step. Weak convergence of (u_k, S_k). The bounds $0 \le u_k, S_k \le 1$, and the elliptic regularity

$$||S_{n_k}||_{W^{1,2}((0,T)\times\Omega)} + ||\nabla S_{n_k}||_{W^{1,2}((0,T)\times\Omega)^N} \le C \quad \forall k \in \mathbb{N}$$

provide us directly with (21), (22) after extracting subsequences.

Second step. The limits u_∞ *and* S_∞ *are independent of time.*

First, since the couple (u_{n_k}, S_{n_k}) is a solution of (1), considering a test function $\varphi = \varphi(t, y)$ in $\mathcal{C}_0^\infty((0,T) \times Y)$, with $\varphi(0, y) = \varphi(T, y) = 0$ for all $y \in \Omega$, then

$$\int_0^T \int_\Omega u_{n_k} \partial_t \varphi = -\int_0^T \int_\Omega \nabla S_{n_k} \cdot \nabla \varphi g(u_{n_k}).$$

The right-hand side goes to 0 as $k \to \infty$ according to (20) for any test function φ. Thus

$$\int_0^T \int_\Omega u_\infty(t, y) \partial_t \varphi(t, y)\, dt\, dy = 0$$

for any test function φ vanishing at $t = 0$ and $t = T$. Consequently, u_∞ is independent of t : $u_\infty(t, y) = u_\infty(y)$.

Furthermore, in the weak limit it holds

$$-\Delta S_\infty + S_\infty = u_\infty \quad \text{on}(0,T) \times \Omega, \qquad \nabla S_\infty \cdot n_\Omega = 0 \quad \text{on } \partial\Omega.$$

and by uniqueness for this problem, we also have $S_\infty = S_\infty(y)$.

Third step. The limiting equation on u_∞.

We now introduce the notations

$$\begin{aligned} A &:= \{(t, y) \in (0,T) \times \Omega;\ \nabla S_{n_k}(t, y) \to 0\}, \\ B &:= \{(t, y) \in (0,T) \times \Omega;\ u_{n_k}(t, y) \to 0\}, \\ C &:= \{(t, y) \in (0,T) \times \Omega;\ u_{n_k}(t, y) \to 1\}. \end{aligned}$$

Then $\lambda\left((0,T) \times \Omega \setminus (A \cup B \cup C)\right) = 0$, where λ is the Lebesgue measure, because from (20), we deduce that

$$\int_0^T \int_\Omega |\nabla S_k|^2(t, y)\, u_k(t, y)(1 - u_k(t, y))\, dt\, dy \to 0$$

as $k \to \infty$. Hence there exists a subsequence, still denoted by n_k, such that

$$|\nabla S_{n_k}|^2(t, y)\, u_{n_k}(t, y)(1 - u_{n_k}(t, y)) \to 0 \quad \text{a.e.}$$

It follows from the above strong convergence results that

$$\int_0^T \int_\Omega \mathbf{1}_A \, |\nabla S_{n_k}|^2 \to 0 = \int_0^T \int_\Omega \mathbf{1}_A \, |\nabla S_\infty|^2 ,$$

$$\int_0^T \int_\Omega \mathbf{1}_B u_{n_k} \to 0 = \int_0^T \int_\Omega \mathbf{1}_B u_\infty,$$

$$\int_0^T \int_\Omega \mathbf{1}_C (1-u_{n_k}) \to 0 = \int_0^T \int_\Omega \mathbf{1}_C (1-u_\infty).$$

Consequently,

$$\int_0^T \int_\Omega |\nabla S_\infty|^2 g(u_\infty) = T \int_\Omega |\nabla S_\infty|^2 g(u_\infty) = 0,$$

and $|\nabla S_\infty|^2 g(u_\infty) = 0$ almost everywhere on Ω. Thus $\nabla S_\infty g(u_\infty) = 0$ and in particular

$$\mathrm{div}_y(\nabla S_\infty \, u_\infty (1-u_\infty)) = 0.$$

Fourth step. Proof of (23).

We have already proved that as $k \to \infty$,

$$\int_0^T \int_\Omega |\nabla S_{n_k}|^2 u_{n_k}(1-u_{n_k}) \to \int_0^T \int_\Omega |\nabla S_\infty|^2 u_\infty (1-u_\infty) = 0.$$

The above convergence results entails that

$$\int_0^T \int_\Omega |\nabla S_{n_k}|^2 u_{n_k}^2 \to \int_0^T \int_\Omega |\nabla S_\infty|^2 u_\infty^2.$$

And since $\nabla S_{n_k} \to \nabla S_\infty$ in $L^2((0,T)\times\Omega)$, it follows that

$$\int_0^T \int_\Omega |\nabla S_\infty|^2 \left(u_{n_k}^2 - u_\infty^2\right) \to 0.$$

Writing

$$(u_{n_k} - u_\infty)^2 = u_{n_k}^2 - u_\infty^2 - 2u_{n_k}u_\infty + 2u_\infty^2,$$

and using once more the weak convergence of u_{n_k}, we obtain

$$\int_0^T \int_\Omega |\nabla S|^2 (u_{n_k} - u_\infty)^2 \to 0.$$

(23) follows easily, extracting a further subsequence if necessary.

Fifth step. Kinetic formulation.

Let $f = f(t, y, \xi)$ be the weak limit of $\mathbf{1}_{\xi < u_{n_k}(t,y)}$. Notice that it is not obvious that f does not depend on t. Then according to the previous steps and to section 3, f satisfies

$$\partial_t f + g(\xi)(\xi - S_\infty)\partial_\xi f + g'(\xi)\nabla S_\infty \cdot \nabla f + g'(\xi)(\rho - u_\infty f) = \partial_\xi m,$$

where m is a nonnegative measure and ρ is related to f by equation (15). Moreover, since $u_{n_k}(t, y)$ converges to $u_\infty(y)$ a.e. on the set $\{y; \nabla S_\infty(y) \neq 0\}$, we deduce that $f(t, y, \xi) = \mathbf{1}_{\xi < u_\infty(y)}$ a.e. on $\{\nabla S_\infty \neq 0\}$, and thus $(\rho - u_\infty f) = 0$ and $g(\xi)\partial_\xi f = 0$ on $\{\nabla S_\infty \neq 0\}$.

□

Remark 5.1. *In general, stationary states of* (1) *are not unique, even when entropy conditions are required and the mean value on* Ω *is prescribed. Thus, it is not obvious that the whole sequence* u_k *should converge to a stationary state* u_∞.

Appendix

This Appendix is devoted to the rigorous proof of inequality (17). Regularizations by convolution are used in order to justify the nonlinear manipulations which led to equation (16), as in [20, 21]. We focus on the case when Ω is an arbitrary bounded domain in $\mathbb{R}^N$, with a $\mathcal{C}^1$ boundary, and $\nabla S \cdot n_\Omega = 0$ on $\partial\Omega$; the case when $\Omega = \Pi_{i=1}^N (0, T_i)$, and S satisfies periodic boundary conditions, is in fact easier, and can be treated in a similar fashion.

We take $\delta_1, \delta_2 > 0$ arbitrary, and $\varphi_1 \in \mathcal{D}(\mathbb{R})$, $\varphi_2 \in \mathcal{D}(\mathbb{R}^N)$, $\varphi_3 \in \mathcal{D}(\mathbb{R})$, with

$$0 \leq \varphi_1, \varphi_1, \varphi_3 \leq 1,$$
$$\int_{\mathbb{R}} \varphi_1 = \int_{\mathbb{R}^N} \varphi_2 = \int_{\mathbb{R}} \varphi_3 = 1,$$
$$\text{Supp}\, \varphi_1 \subset [-1, 0], \qquad \text{Supp}\, \varphi_2 \subset B_1, \qquad \text{Supp}\, \varphi_3 \subset [-1, 1].$$

We set $\delta = (\delta_1, \delta_2)$, and

$$\varphi_\delta(t, y, \xi) = \frac{1}{\delta_1\, \delta_2^{N+1}} \varphi_1\left(\frac{t}{\delta_1}\right)\, \varphi_2\left(\frac{y}{\delta_2}\right)\, \varphi_3\left(\frac{\xi}{\delta_2}\right),$$

and for $(t, y, \xi) \in [0, \infty) \times \mathbb{R}^{N+1}$

$$f_\delta(t, y, \xi) := f * \varphi_\delta(t, y, \xi) = \int_{\mathbb{R}} \int_{\Omega} \int_{\mathbb{R}} f_\delta(t', y', \xi') \varphi_\delta(t - t', y - y', \xi - \xi') dt' \, dy' \, d\xi',$$
$$m_\delta := m * \varphi_\delta.$$

Then f_δ and m_δ are smooth functions of t, y, ξ for all $\delta > 0$, and $0 \leq f_\delta \leq 1$, $m_\delta \geq 0$. Moreover, f_δ is a solution of

$$\partial_t f_\delta + (\xi - S) g(\xi) \partial_\xi f_\delta + g'(\xi) \nabla_y S \cdot \nabla_y f_\delta + R * \varphi_\delta = \partial_\xi m_\delta + r_\delta, \qquad (24)$$

and the remainder r_δ is equal to

$$\begin{aligned} r_\delta &= (\xi - S) g(\xi) \partial_\xi f_\delta - [(\xi - S) g(\xi) \partial_\xi f] * \varphi_\delta \\ &\quad + g'(\xi) \nabla_y S \cdot \nabla_y f_\delta - \left[g'(\xi) \nabla_y S \cdot \nabla_y f \right] * \varphi_\delta. \end{aligned}$$

We wish to stress that equation (24) holds everywhere in $(0, \infty) \times \Omega_\delta \times \mathbb{R}$, and not in $(0, \infty) \times \Omega \times \mathbb{R}$, where

$$\Omega_\delta := \{ y \in \Omega, \ d(y, \partial\Omega) \geq \delta \}.$$

This yields a small difficulty when integrating equation (24) on $(0, \infty) \times \Omega_\delta \times \mathbb{R}$, because $\nabla S \cdot n_{\Omega_\delta}(y) \neq 0$ on $\partial\Omega_\delta$ even though $\nabla S \cdot n_\Omega(y) = 0$ on $\partial\Omega$. However, this difficulty can be overcome by using the regularity of S and of the boundary $\partial\Omega$.

Before writing an equation for $f_\delta - f_\delta^2$, let us first prove that $r_\delta \to 0$ in $L^1((0, T) \times \Omega \times (-R, R))$ for all $T, R > 0$. In the rest of the appendix, we set $z = (t, y, \xi) \in \mathbb{R}^{N+2}$, with $z_0 = t$, $z_i = y_i$ for $1 \leq i \leq N$, $z_{N+1} = \xi$. Accordingly, we define the differential operators

$$\partial_0 = \frac{\partial}{\partial t}, \ \partial_i = \frac{\partial}{\partial y_i} \quad 1 \leq i \leq N, \ \partial_{N+1} = \frac{\partial}{\partial \xi}$$

and we set $Q := (0, \infty) \times \Omega \times \mathbb{R}$.

Then for instance, we have

$$\begin{aligned} & (\xi - S) g(\xi) \partial_\xi f_\delta - [(\xi - S) g(\xi) \partial_\xi f] * \varphi_\delta \\ = \; & (\xi - S) g(\xi) f * \partial_\xi \varphi_\delta - [(\xi - S) g(\xi) f] * \partial_\xi \varphi_\delta \\ & + [\partial_\xi \left((\xi - S) g(\xi) \right) f] * \varphi_\delta \\ = \; & \int_Q \left(G(z) - G(z') \right) f(z') \partial_{N+1} \varphi_\delta(z - z') \, dz' \\ & + [\partial_{N+1} G \, f] * \varphi_\delta \end{aligned}$$

where

$$G(z) := (\xi - S(t,y))g(\xi), \quad z = (t,y,\xi).$$

Then $[\partial_{N+1}G\, f] * \varphi_\delta$ converges to $\partial_{N+1}G\, f$ in $L^p((0,T)\times\Omega\times(R,R))$ for all $T,R>0$. And setting $\psi_k(z) = z_k\partial_{N+1}\varphi(z)$ for $1\le k\le N$, we have

$$\begin{aligned}
&\int_Q \left(G(z)-G(z')\right) f(z')\partial_{N+1}\varphi_\delta(z-z')\,dz' \\
=\ &\int_Q\int_0^1 \partial_k G(\tau z + (1-\tau)z')\, f(z')\psi_{k,\delta}\left(z-z'\right)\ dz'\,d\tau
\end{aligned}$$

The above integral converges to

$$\partial_k G(z) f(z)\int_{\mathbb{R}^{N+2}}\psi_k(z')\,dz'$$

in $L^2((0,T)\times\Omega\times(-R,R))$ for all $T,R>0$ (recall that S is bounded in $L^\infty(0,T;W^{2,q}(\Omega))\cap W^{1,2}(0,T;H^1(\Omega))$ thanks to proposition 3.1). But $\int_{\mathbb{R}^{N+2}}\psi_k(z')\,dz' = 0$ if $k\ne N+1$ and $\int_{\mathbb{R}^{N+2}}\psi_{N+1}(z')\,dz' = -1$. Thus

$$(\xi - S)g(\xi)\partial_\xi f_\delta - [(\xi-S)g(\xi)\partial_\xi f]*\varphi_\delta$$

converges to 0 in $L^2((0,T)\times\Omega\times(-R,R))$ for all $T,R>0$. The other term can be treated in a similar way, using the bounds on S derived in proposition 3.1.

We now go back to the equation on f_δ; since f_δ is smooth in t,y,ξ, we can use the chain rule and write, for $(t,y,\xi)\in(0,\infty)\times\Omega_\delta\times\mathbb{R}$,

$$\begin{aligned}
\partial_t\left(f_\delta - f_\delta^2\right) + (\xi - S)g(\xi)\partial_\xi\left(f_\delta - f_\delta^2\right) + g'(\xi)\nabla_y S\cdot\nabla_y\left(f_\delta - f_\delta^2\right) + \\
+ R*\varphi_\delta(1-2f_\delta) = \partial_\xi m_\delta(1-2f_\delta) + r_\delta(1-2f_\delta)
\end{aligned}$$

We now integrate the above equation on $\Omega_\delta\times\mathbb{R}$; notice that since $f=0$ for $\xi>1$ and $f=1$ for $\xi<0$, we have $f_\delta - f_\delta^2 = 0$ for $\xi\le-\delta$ or $\xi\ge 1+\delta$, and similarly, $m_\delta, r_\delta = 0$ for $\xi \le -\delta$ or $\xi \ge 1+\delta$. Thus

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_\delta\times\mathbb{R}} \left(f_\delta - f_\delta^2\right) &= \int_{\Omega_\delta\times\mathbb{R}} \left(f_\delta - f_\delta^2\right)\left[\partial_\xi\left((\xi - S)g(\xi)\right) + g'(\xi)\Delta S\right] \\
&\quad - \int_{\Omega_\delta\times\mathbb{R}} R * \varphi_\delta (1 - 2f_\delta) \\
&\quad + 2\int_{\Omega_\delta\times\mathbb{R}} m_\delta \partial_\xi f_\delta + \int_{\Omega_\delta\times\mathbb{R}} r_\delta(1 - 2f_\delta) \\
&\quad - \int_{\partial\Omega_\delta\times\mathbb{R}} \left(f_\delta - f_\delta^2\right) g'(\xi)\nabla S\cdot n_{\Omega_\delta}(y)\, dS(y) d\xi \\
&\le C\int_{\Omega_\delta\times\mathbb{R}} \left(f_\delta - f_\delta^2\right) + C\int_{\Omega_\delta\times\mathbb{R}} \left|f - f^2\right| * \varphi_\delta \\
&\quad + ||r_\delta(t)||_{L^1(\Omega\times(-1,2))} + C\int_{\partial\Omega_\delta} \left|\nabla S\cdot n_{\Omega_\delta}(y)\right|\, dS(y)
\end{aligned}$$

In the above inequality, we have used the fact that $\partial_\xi f_\delta = -\nu * \varphi_\delta \le 0$, where ν was defined in (14), together with lemma 3.1. Moreover, notice that since the function $x \mapsto x - x^2$ is concave, by Jensen's inequality, we get

$$\int_{\Omega_\delta\times\mathbb{R}} \left(f - f^2\right) * \varphi_\delta \le \int_{\Omega_\delta\times\mathbb{R}} \left(f_\delta - f_\delta^2\right).$$

And since S belongs to $W^{2,q}$ for all $q < \infty$, $\nabla S \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$; remember that we have assumed that the boundary $\partial\Omega$ is at least $\mathcal{C}^1$. In such conditions, it is easily proved that

$$\int_{\partial\Omega_\delta} \left|\nabla S\cdot n_{\Omega_\delta}(y)\right|\, dS(y) \to 0$$

as $\delta \to 0$ for almost every $t > 0$. In the following, we set

$$U_\delta(t) = ||r_\delta(t)||_{L^1(\Omega\times(-1,2))} + C\int_{\partial\Omega_\delta} \left|\nabla S\cdot n_{\Omega_\delta}(y)\right|\, dS(y),$$

and we have proved that $U_\delta \to 0$ in $L^1_{\text{loc}}([0,\infty))$ as $\delta \to 0$.

Thus we are led to

$$\frac{d}{dt}\int_{\Omega_\delta\times\mathbb{R}} \left(f_\delta - f_\delta^2\right) \le C\int_{\Omega_\delta\times\mathbb{R}} \left(f_\delta - f_\delta^2\right) + U_\delta(t).$$

Consequently, by Gronwall's lemma,

$$\begin{aligned}
\int_{\Omega_\delta\times\mathbb{R}} \left(f_\delta(t) - f_\delta(t)^2\right) &\le e^{Ct}\int_{\Omega_\delta\times\mathbb{R}} \left(f_\delta(t=0) - f_\delta(t=0)^2\right) \\
&\quad + \int_0^t e^{C(t-s)} U_\delta(s)\, ds. \qquad (25)
\end{aligned}$$

There only remains to prove that $f_\delta(t=0) - f_\delta(t=0)^2$ goes to 0 as $\delta \to 0$. This is a consequence of the fact that $f_\delta(t=0)$ strongly converges to $f(t=0) = \mathbf{1}_{\xi < u_0}$, but the latter is not obvious since

$$f_\delta(t=0,y,\xi) = \int_0^\infty \int_{\Omega\times\mathbb{R}} f(t',y',\xi')\varphi_\delta(-t',y-y',\xi-\xi')\,dt'\,dy'\,d\xi'.$$

We therefore use the same technique as in [21], Lemma 4.2.2: since the proof is strictly identical to the one in [21], we only recall briefly the main arguments.

Let

$$T_{\delta_1}(t) = 1 - \frac{1}{\delta_1}\int_0^t \varphi_1\left(-\frac{s}{\delta_1}\right)\,ds;$$

then $\partial_t T_{\delta_1}(t) = -\frac{1}{\delta_1}\varphi_1\left(-\frac{t}{\delta_1}\right)$, and thus for $(t,y,\xi) \in (0,\infty)\times\Omega_\delta\times\mathbb{R}$, $f_\delta(t=0,y,\xi)$ can be written as

$$\begin{aligned}
& f_\delta(t=0,y,\xi)\\
= \; & I_\delta(y,\xi)\\
& +\partial_\xi \int_0^\infty\int_{\mathbb{R}^{N+1}} m(t',y',\xi')\frac{1}{\delta_2^{N+1}}\varphi_2\left(\frac{y-y'}{\delta_2}\right)\varphi_3\left(\frac{\xi-\xi'}{\delta_2}\right)\,dt'\,dy'\,d\xi'\\
& + \int_{\mathbb{R}^{N+1}} \mathbf{1}_{\xi<u_0(y)}\frac{1}{\delta_2^{N+1}}\varphi_2\left(\frac{y-y'}{\delta_2}\right)\varphi_3\left(\frac{\xi-\xi'}{\delta_2}\right)\,dy'\,d\xi'
\end{aligned}$$

where $||I_\delta||_{L^\infty} \le C\delta_1/\delta_2$. Passing first to the weak limit as $\delta_1,\delta_2 \to 0$ with $\delta_1/\delta_2 \to 0$ in the above equation entails that the weak limit F of $f_\delta(t=0,y,\xi)$ satisfies

$$F = \partial_\xi M + 1_{\xi<u_0(y)}$$

for some nonnegative measure M vanishing for large ξ. This leads to $F = 1_{\xi<u_0(y)}$ and $M=0$ thanks to a lemma in [21]. Then, the above formula for $f_\delta(t=0,y,\xi)$ is used once again to find the weak limit of $f_\delta(t=0,y,\xi)^2$. It is easily proved that

$$f_\delta(t=0,y,\xi)^2 \rightharpoonup F^2 = 1_{\xi<u_0(y)},$$

and thus the convergence is strong.

Consequently, $f_\delta - f_\delta^2$ converges to 0 in $L^1_{\text{loc}}((0,\infty)\times\Omega\times\mathbb{R})$. Since $f_\delta - f_\delta^2 \to f - f^2$ in $L^1_{\text{loc}}(0,\infty;L^1(\Omega\times\mathbb{R}))$, we deduce that $f=0$ or $f=1$ almost everywhere. The rest of the proof, exposed in section 4, is therefore justified.

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